IOSR Journal Of Applied Physics (IOSR-JAP)
e-ISSN: 2278-4861.Volume 16, Issue 1 Ser. I (Jan. – Feb. 2024), PP 43-47
www.Iosrjournals.Org

The Solution Of Exponential Forced Sine Gordon Equation By Modified Homotopy Perturbation Method

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Abstract
In this paper we try to find the approximate solution of Exponential Forced Sine Gordon equation by the Modified Homotopy perturbation method. We formulate our equation by using the modified homotopy perturbation method so as to construct our homotopy equations, which can be solved by a normal integration. Then the summations of these solutions give us the final solution of our equation. We found our graphs solutions by using Mathematica computer programs which give us a positive and negative soliton solution.
Keywords: *Exponential Forced Sine Gordon equation, Modified homotopy Perturbation,*

Date of Submission: 07-01-2024 Date of Acceptance: 17-01-2024

I. Introduction

In this Section we talk about the Modified Homotopy Perturbation Method. In general the modified Homotopy Perturbation Method proposed for solving the nonlinear differential equation is based upon a small parameter with the Homotopy method one (He, J., 1999 a). our process for approximation solution begin after substituting the assumed approximation solution into the homotopy and solving our corresponding equations,**.**

Lu, J. (2009), he proposed the Modified Homotopy Perturbation Method for solving sine Gordon equation without forcing terms. But in our paper we need to consider the forced terms one. **Lu, J. (2009)**, in his work introduced a small parameter and Taylor series expansion to modify the homotopy perturbation method. After that he gets a new analytical approach for solving the initial value problem for the following sine Gordon Equation,

$$u_{tt} - u_{xx} + \sin u = 0 \qquad (1)$$

Subject to the initial conditions $u(x,t_0) = g_1(x) \qquad u_t(x,t_0) = g_2(x)$

The main feature of his method from the old one has ability to obtain the analytical or approximate solution to the Sine Gordon Equation without linearization or discretization process so as to avoid the difficulties of calculation which is involved or appears in the polynomials in the old method such as Adomain decomposition or the old homotopy method.

II. Approximation Solution without Forcing Terms

In this case, it is clear that we also faced equation involve **sine u**, this makes it very complicated to get the solution. To avoid this difficulty, we consider another approach for dealing and solving this equation, on the basis of the homotopy perturbation method. We introduce a variable parameter $p \in [0,1]$ in the Sine Gordon equation. Then equation (1) becomes,

$$u_{tt} - pu_{xx} + \sin pu = 0 \qquad (2)$$

Subject to the initial conditions $u(x,t_0) = g_1(x) \qquad u_t(x,t_0) = g_2(x)$

It is easy to see that when **p = 0**, equation (2) corresponds to a linear equation, and while when **p = 1**, it is just the original nonlinear one. The parameter **p** is introduced naturally and not affected by artificial factors. Furthermore, it can be considered as a small parameter.

By applying the modified homotopy perturbation technique, we assume that the solution of equation (2), can be expressed in terms of **p** as follows,

$$u = \sum_{n=0}^{\infty} p^n u_n = u_0 + pu_1 + p^2 u_2 + p^3 u_3 + ... \qquad (3)$$

Substituting p = 1, the results is the approximate solution of equation (1).

$$u = \lim_{p\to 1} u \quad \sum_{n=0}^{\infty} u_n = u_0 + u_1 + u_2 + u_3 + ... \qquad (4)$$

To obtain the approximate solution of equation (2), we consider the Taylor series expansion of **sin u** in the following form,

$$\sin u = u - \frac{u^3}{3!} + \frac{u^5}{5!} - ....(-1)^n \frac{u^{2n-1}}{(2n-1)!} + O(n+1) \qquad (5)$$

Substituting equation (3) and (5) into (2) and comparing the coefficients of identical degrees of **p**. Then we get the following equations.

$$p^0 \quad : u_{0tt} = 0, \quad u_0(x,t_0) = g_1(x), u_{0t}(x,t_0) = g_2(x)$$

$$p^1 \quad : u_{1tt} - u_{0xx} + u_0 = 0, \quad u_1(x,t_0) = 0, u_{1t}(x,t_0) = 0$$

$$p^2 \quad : u_{2tt} - u_{1xx} + u_1 = 0, \quad u_2(x,t_0) = 0, u_{2t}(x,t_0) = 0$$

$$p^3 \quad : u_{3tt} - u_{2xx} + u_2 - \frac{u_0^3}{3!} = 0, \quad u_3(x,t_0) = 0, u_{3t}(x,t_0) = 0\,,$$

and so on. Then we solve the above equations by simple integration, we found the values of u_0, u_1, u_2, u_3, u_4 and so on. After that we can get the approximate solution of equation (1) as follows

$$u = u_0 + u_1 + u_2 + u_3 + ...u_n \qquad (6)$$

Then we can obtain the **n^th^** order approximate solutions .It is clear that and very easy to calculate more components to improve our final solutions

III. Approximation Solution of our Model with Forcing terms

In this Section we need to solve our model of equation (7) with Exponential forcing terms and the following initial conditions by Modified Homotopy Perturbation Method,

$$u_{tt} - u_{xx} + \sin u = e^x \qquad (7)$$

Subject to the initial conditions,

$$u(x,0) = 0 \qquad u_t(x,0) = 4\sec hx.$$

In this case the forcing terms which is equal e^x in the form of positive exponential and called the nonlocal forcing terms. Then by modified homotopy perturbation method, we constructed the following homotopy and then the equation (7) becomes,

$$u_{tt} - pu_{xx} + \sin pu - pe^x = 0 \qquad (8)$$

Subject to the initial conditions, $u(x,0) = 0 \qquad u_t(x,0) = 4\sec hx$ and where **p** is small parameter belong to [0, 1], and by applying the modified homotopy perturbation technique, we assume that the solution of equation (8) can be expressed in terms of **p** as follows,

$$u = \sum_{n=0}^{\infty} p^n u_n = u_0 + pu_1 + p^2 u_2 + p^3 u_3 + ... \qquad (9)$$

Substituting p = 1 in the above equation, the results is the approximate solution of equation (7).

$$u = \sum_{n=0}^{\infty} u_n = u_0 + u_1 + u_2 + u_3 + ... \qquad (10)$$

To obtain the approximate solution of equation (7), we consider the Taylor series expansion of **sin u** in the following form,

$$\sin u = u - \frac{u^3}{3!} + \frac{u^5}{5!} - ....(-1)^n \frac{u^{2n-1}}{(2n-1)!} + O(n+1) \qquad (11)$$

Substituting equation (9) and equation (11) into equation (7) and then by comparing the coefficients of identical degrees of p, we will get the following equations,

$$p^0 \quad : u_{0tt} = 0, \quad u_0(x,0) = 0, u_{0t}(x,0) = 4\sec h\, x$$
$$p^1 \quad : u_{1tt} - u_{0xx} + u_0 - e^x = 0, \quad u_1(x,0) = 0, u_{1t}(x,0) = 0$$

$$p^2 \quad : u_{2tt} - u_{1xx} + u_1 = 0, \quad u_2(x,0) = 0, u_{2t}(x,0) = 0$$

$$p^3 \quad : u_{3tt} - u_{2xx} + u_2 - \frac{u_0^3}{3!} = 0, \quad u_3(x,0) = 0, u_{3t}(x,0) = 0$$
$$and \quad so\ on$$

Solving the above equations by simple integration, we get following results,

For u_0 values we get the following,

$$u_0 \;= g_1(x) + t\, g_2(x) = 0 + t4\sec hx = 4t\sec hx$$
$$u_0' \;= -4t\sec hx\tanh x$$
$$u_0'' = -4t\left(\sec hx.\sec h^2 x + \tanh . - \sec hx\tanh x\right) = -4t\sec h^3 x + 4t\sec hx\tanh^2 x$$
$$we \quad know\ that \quad \tanh^2 x = 1 - \sec h^2 x \quad sub.\ in \quad the\ above\ equation\ we\ get$$
$$u_0'' = -4t\sec h^3 x + 4t\sec hx\left(1 - \sec h^2 x\right) - 4t\sec h^3 x + 4t\sec hx - 4t\sec h^3 x$$
$$u_0'' = -8t\sec h^3 x + 4t\sec hx$$
$$then\ we\ needs\ to\ find \quad u_1 \quad = \quad \int_0^t\int_0^t \left(u_0'' - u_0' - e^x\right) dt\ dt\ then$$
$$\int_0^t\int_0^t\left(-8t\sec h^3 x + 4t\sec hx - 4t\sec hx\right) dt\ dt = \int_0^t\int_0^t\left(-8t\sec h^3 x - 1\right) dt dt = -\frac{8t^3}{6}\sec h^3 x = -\frac{4}{3}t^3\sec h^3 x - \frac{t^2}{2} \quad then$$
$$u_1 \;= -\frac{4t^3}{3}\sec h^3 x - \frac{t^2}{2}e^x$$
$$by\ the\ same\ method\ , we\ get$$
$$u_2 = \frac{4t^5}{15}\left(2 - \cosh x\right)\left(\sec h^5 x\right)$$
$$u_3 = \frac{8t^5}{15}\sec h^3 x - \frac{t^7}{315}\left(96 - 80\cosh 2x\right) + 4\cosh 4x.$$

In the same way, more components can be calculated. Then the approximation solution of our equation (7) for 3^{rd} order can be get it in the following form.

$$u(x,t) = 4t\sec hx - \frac{4t^3}{3}\sec h^3 x - \frac{t^2}{2}e^x + \frac{4t^5}{15}\left(2 - \cosh x\right)\left(\sec h^5 x\right) + \frac{8t^5}{15}\sec h^3 x - \frac{t^7}{315}\left(96 - 80\cosh 2x\right) + 4\cosh 4x.$$

All our graphs approximation solutions were found by using Mathematica Computer Programs. All our graphs showed that the solutions in time t equal 0.5

The Figures Bellows shows the solutions of our equation (7) with t values equal 0.5 and differences values of x so as to compare between the solutions.

Figure: 1 gives the approximate solution of our model of equation **(7)** with t values equal **0.5** and x between **-50 and 50**.

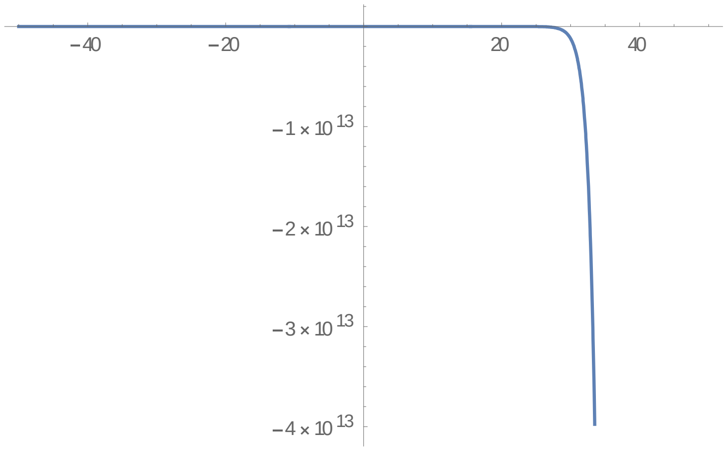

Figure: 1

Figure: 2 give the approximate solution of our model equation **(7)** with **t** values equal **0.5** and x between **-5 and 5**

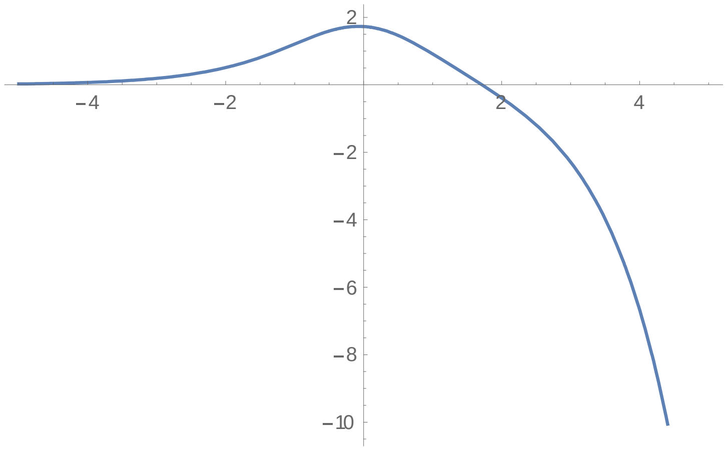

Figure: 2

Figure: 3 give the approximate solution of our model equation (7) with **t** values equal **0.5** and x between **-10** and **10**.

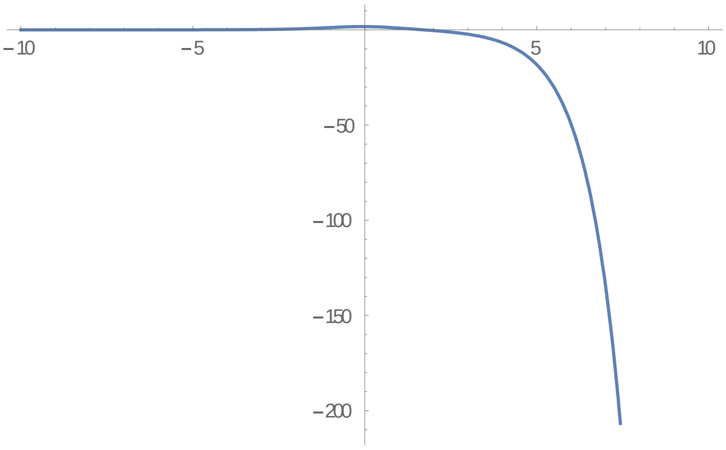

Figure: 3

Figure: 4 give the approximate solution of our model equation **(7)** with t values equal 0.5 and x between -8 and 2.

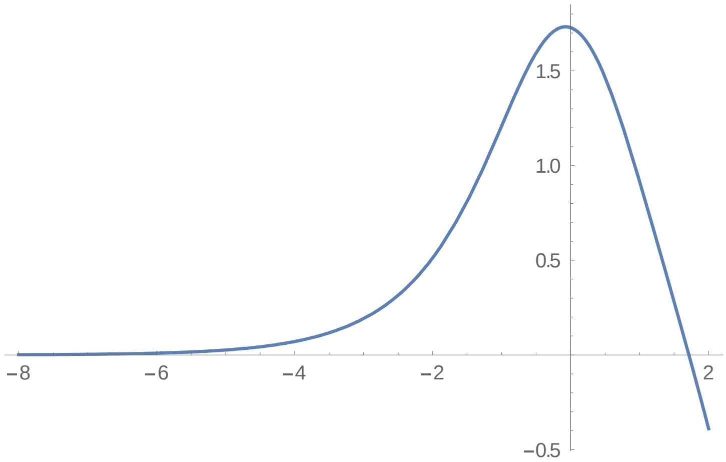

Figure: 4

All graph solutions of Figure **No. 2 and 4** give us full shape of soliton with positive and negative. **Figure No. 1 and 3** gives us only negatives and not completed soliton shapes

IV. Conclusion

In this paper we studied the approximation solution of Exponential Forced Sine Gordon equation by the modified homotopy perturbation method. We found that some of these solutions give us complete form of soliton solutions in positive and negative areas and rest solutions one give us uncompleted shape of Soliton with negative. The future work will be for another types of Forced terms of Sine Gordon equation one.

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