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# Radial Free Fall into Schwarzschild Black Holes Using the Generalized Metric

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***Abstract***
*The problem of radial free fall one of the most interesting problems in black holes physics. Using the Generalized Metricusing the velocity and acceleration equations of radial free falling massive particle has been formulated. Applying weak field approximation of generalized equations of velocity and acceleration,the same GR equations for acceleration and velocity obtained*

***Key words:*** *Generalized Metric,Schwarzschild Black Holes,General relativity (GR), generalized field equation (GFE)*

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## I. Introduction

Einstein general relativity, through its geometricalstructure of spacetime, solved most of the problems of Newtonian mechanics.In spite of these successes GRsuffers from noticeable setbacks.For instancethe nature of some astronomical exoticaincluding quasars and pulsars, where the strong field is presumed to be predominant,is difficult to be understood in terms of GR [2].

Many attempts were made to go beyond GR [3]. Among these the generalized FieldEquation (GFE), which is based on a more general form of fourth order differential equations,is the best candidate. This is due to the successes of higher order theories in makingthese problems controllable [4,5,6,7]

In this paper, we used generalized metric [1] instead of Schwarzschild metric, which is seems convenient for strong field as GFE predicted [8]. In section2 we set the geodesic equations for massive particle which is travel along timelike worldline In section3 we set the four velocity equation in order to calculate the radial velocity.

Insection4 we considered the case of no angular momentum in order to get pure radial acceleration of massive particle and get an answer what is the velocity of the particle when it hit the physical singularity of the black hole

### 1. The Geodesic Equation of Massive Particle

Massive particle travel along timelike worldline which means that the square of the magnitude of it four velocity given by

$$g_{\mu\nu} u^\mu u^\nu = u^\mu u_\mu = -1 \tag{1}$$

The condition of parallel transport is

$$\frac{d}{d\lambda}\left(u^\mu \vec{e}_\mu\right) = 0 \tag{2}$$

$$\frac{du^\mu}{d\lambda}\vec{e}_\mu + \frac{d\vec{e}_\mu}{d\lambda}u^\mu = \frac{du^\mu}{d\lambda}\vec{e}_\mu + u^\mu \Gamma^\beta_{\mu\nu}\vec{e}_\beta \frac{dx^\nu}{d\lambda} = 0 \tag{3}$$

$$\left[\frac{d}{d\lambda}\left(\frac{dx^\mu}{d\lambda}\right) + u^\beta \Gamma^\mu_{\beta\nu} \quad \frac{dx^\nu}{d\lambda}\right]\vec{e}_\mu = 0 \tag{4}$$

Then the geodesic equations given by

$$\frac{d^2 x^\mu}{d\lambda^2} + u^\beta \Gamma^\mu_{\beta\nu} \quad \frac{dx^\nu}{d\lambda} = 0 \tag{5}$$

Now we are going to write the geodesic equation in generalized metric [1]to be static and takes the form

$$g_{\mu\nu} x^\mu x^\nu = -e^{-2m/r}dt^2 + e^{2m/r}dr^2 + r^2 d\theta^2 + r^2 sin^2\theta d\varphi^2 \tag{6}$$

Where $A(r) = e^{2m/r}$ and $B(r) = e^{-2m/r}$
According to this the Christoffel symbols for such metric is given by

$$\Gamma_{11}^{1} = \frac{A'(r)}{2A(r)} \Gamma_{22}^{1} = \frac{-r}{2A(r)} \Gamma_{33}^{1} = \frac{-r\sin^2\theta}{2A(r)} \Gamma_{00}^{1} = \frac{B'(r)}{2A(r)} = \frac{m}{r^2}e^{-4m/r} \Gamma_{12}^{2} = \Gamma_{21}^{2} = \frac{1}{r}$$
$$\Gamma_{33}^{2} = \sin\theta\cos\theta\, \Gamma_{13}^{3} = \Gamma_{31}^{3} = \frac{1}{r} \Gamma_{23}^{3} = \Gamma_{32}^{3} = \cot\theta\, \Gamma_{10}^{0} = \Gamma_{01}^{0} = \frac{B'(r)}{2B(r)} \tag{7}$$

To find the four geodesic equations we substitute Christoffel symbols in equation (5)

$$\frac{d^2t}{d\lambda^2} + \Gamma_{10}^{0} \frac{dr}{d\lambda}\frac{dt}{d\lambda} + \Gamma_{01}^{0} \frac{dt}{d\lambda}\frac{dr}{d\lambda} = 0 \tag{8}$$
$$\frac{d^2r}{d\lambda^2} + \Gamma_{11}^{1} \frac{dr}{d\lambda}\frac{dr}{d\lambda} + \Gamma_{22}^{1} \frac{d\theta}{d\lambda}\frac{d\theta}{d\lambda} + \Gamma_{33}^{1} \frac{d\varphi}{d\lambda}\frac{d\varphi}{d\lambda} + \Gamma_{00}^{1} \frac{dt}{d\lambda}\frac{dt}{d\lambda} = 0 \tag{9}$$
$$\frac{d^2\theta}{d\lambda^2} + \Gamma_{12}^{2} \frac{dr}{d\lambda}\frac{d\theta}{d\lambda} + \Gamma_{21}^{2} \frac{dr}{d\lambda}\frac{d\theta}{d\lambda} + \Gamma_{33}^{2} \left(\frac{d\varphi}{d\lambda}\right)^2 = 0 \tag{10}$$
$$\frac{d^2\varphi}{d\lambda^2} + \Gamma_{13}^{3} \frac{d\varphi}{d\lambda}\frac{dr}{d\lambda} + \Gamma_{31}^{3} \frac{d\varphi}{d\lambda}\frac{dr}{d\lambda} + \Gamma_{23}^{3} \frac{d\theta}{d\lambda}\frac{d\varphi}{d\lambda} + \Gamma_{32}^{3} \frac{d\theta}{d\lambda}\frac{d\varphi}{d\lambda} = 0 \tag{11}$$

Applying the Christoffel symbols in equation (7) one can generalize geodesics to fit with the strong gravitational field

$$\frac{d^2t}{d\lambda^2} + \frac{2m}{r^2} \frac{dr}{d\lambda}\frac{dt}{d\lambda} = 0 \tag{12}$$
$$\frac{d^2r}{d\lambda^2} - \frac{m}{r^2}\left(\frac{dr}{d\lambda}\right)^2 - \frac{r}{2}e^{-2m/r}\left(\frac{d\theta}{d\lambda}\right)^2 - \frac{r sin^2\theta}{2}e^{-2m/r}\left(\frac{d\varphi}{d\lambda}\right)^2 + \frac{m}{r^2}e^{-4m/r}\left(\frac{dt}{d\lambda}\right)^2 = 0 \tag{13}$$
$$\frac{d^2\theta}{d\lambda^2} + \frac{2}{r}\frac{dr}{d\lambda}\frac{d\theta}{d\lambda} - \sin\theta\cos\theta\left(\frac{d\varphi}{d\lambda}\right)^2 = 0 \tag{14}$$
$$\frac{d^2\varphi}{d\lambda^2} + \frac{2}{r}\frac{d\varphi}{d\lambda}\frac{dr}{d\lambda} + 2\cot\theta\frac{d\theta}{d\lambda}\frac{d\varphi}{d\lambda} = 0 \tag{15}$$

The solution of these equations depend mainly on initial condition,therefore we can use equation (13) to find the radial free fall equation by eliminating the other components of motion

## 2. Generalized Four Velocity

The magnitude of the four velocity of massive particle is constant along geodesic (1). we have to exchange $\lambda$ by $\tau$therefore

$$g_{\mu\nu}u^{\mu}u^{\nu} = u^{\mu}u_{\mu} = -1 \tag{16}$$
$$g_{\mu\nu}u^{\mu}u^{\nu} = -e^{-2m/r}\left(\frac{dt}{d\tau}\right)^2 + e^{2m/r}\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\theta}{d\tau}\right)^2 + r^2\sin^2\theta\left(\frac{d\varphi}{d\tau}\right)^2 = -1 \tag{17}$$

We know the Schwarzschild metric is independent of time and so there exists a killing vector associated with invariance under time translations and the result of this conservation of energy applying to our massive particle so the killing vector is

$$k^0 = (1,0,0,0)k_0 = g_{00}k^0 \tag{18}$$

That obey the condition

$$\frac{d}{d\tau}(k^t.\vec{u}) = 0 \tag{19}$$

Where $\frac{p}{m_0} = \frac{m_0\vec{u}}{m_0} = \vec{u}$is momentum per unit mass.
Therefore,the derivative (19) takes the form

$$\frac{d}{d\tau}(k^t.\vec{u}) = \frac{d}{d\tau}(g_{00}k^0u^0) = \frac{d}{d\tau}\left(g_{00}\frac{p}{m_0}\right) = \frac{d}{d\tau}\left(e^{-2m/r}\frac{dt}{d\tau}\right) = 0 \tag{20}$$

Butting

$$e^{-2m/r}\frac{dt}{d\tau} = \varepsilon = const \tag{21}$$

That means

$$\left(\frac{dt}{d\tau}\right)^2 = \left(e^{-2m/r}\right)^{-2}\varepsilon^2 \tag{22}$$

Where $\varepsilon = \frac{E}{m_0}$represent the total energy per unit mass. Now one can use killing vector [10]

$$k^{\varphi} = (0,0,0,1)k_{\varphi} = g_{\varphi\varphi}k^{\varphi} \tag{23}$$

That obey the condition

$$\frac{d}{d\tau}\left(g_{\varphi\varphi}\,k^{\varphi}\right)=0 \tag{24}$$

Therefore, we have

$$\frac{d}{d\tau}\left(g_{\varphi\varphi}\,k^{\varphi}.m_0u^{\varphi}\right)=\frac{d}{d\tau}\left(r^2\sin^2\theta\,.m_0\frac{d\varphi}{d\tau}\right)=0 \tag{25}$$

$$l=\frac{L}{m_0}=r^2\sin^2\theta\frac{d\varphi}{d\tau}=const \tag{26}$$

That means

$$\left(\frac{d\varphi}{d\tau}\right)^2=\frac{l^2}{r^4\sin^4\theta} \tag{27}$$

Substituting (22) and (27) into (17) and $\sin\theta=\frac{\pi}{2}$we get

$$\left(\frac{dr}{d\tau}\right)^2=\varepsilon^2-\left(\frac{l^2}{r^2}+1\right)e^{-2m/r} \tag{28}$$

### 3. Generalized Radial Free Fall Acceleration

For the case of radial motion $\varphi$ with not change as a function of proper time $\tau$ so

$$\left(\frac{d\varphi}{d\tau}\right)=0 \qquad l=0 \tag{29}$$

Thusequation (28) reduce to

$$\left(\frac{dr}{d\tau}\right)^2=\varepsilon^2-e^{-2m/r} \tag{30}$$

This equation represents the radial velocity as a function of distance from black hole singularity it is easy to see that the particle hit the singularity with velocity $\frac{dr}{d\tau}=\varepsilon$ when $r=0$

One can differentiate both side of equation with respect to proper time $\tau$

$$2\frac{dr}{d\tau}\frac{d^2r}{d\tau^2}=\frac{d}{d\tau}\left(\varepsilon^2-e^{-2m/r}\right) \tag{31}$$

$$2\frac{dr}{d\tau}\frac{d^2r}{d\tau^2}=-\frac{2me^{-2m/r}}{r^2}\frac{dr}{d\tau} \tag{32}$$

Finally, we get the acceleration of free falling massive particle in the radial direction in generalized Schwarzschild spacetime

$$\frac{d^2r}{d\tau^2}=-\frac{me^{-2m/r}}{r^2} \tag{33}$$

### 4. Weak field approximation

The weak field mean that$\frac{2m}{r}$ small enough to satisfy the following approximation

$$e^{-2m/r}\to 1-\frac{2m}{r}+\frac{2m^2}{r^2}\dots \tag{34}$$

Thus, equation (30) takes the form

$$\left(\frac{dr}{d\tau}\right)^2=\varepsilon^2-\left(1-\frac{2m}{r}+\frac{2m^2}{r^2}\dots\right) \tag{35}$$

For the case of weak field when $r\gg m$Equ (35) reduce to same formula of GR of velocity

$$\left(\frac{dr}{d\tau}\right)^2\approx\varepsilon^2-1+\frac{2m}{r} \tag{36}$$

On other hand by substituting (34) in (31) one gets

$$\frac{d^2r}{d\tau^2}=-\frac{m}{r^2}+\frac{2m^2}{r^3}\dots. \tag{37}$$

Forweak field when$r\gg m$,equ (37) reduce to same equation of acceleration that general relativity predicted

$$\frac{d^2r}{d\tau^2}\approx-\frac{m}{r^2} \tag{38}$$

## II. Conclusion and Discussion

Equation (30) represent the velocity of radially free falling particle into black hole, one can calculate the velocity of particle when it hits the singularity at $r=0$, whileEqu (36) of velocity according to GR gives infinity value for $r\to 0$. Equ (32) represents the acceleration of free falling massive particle in the radial direction in generalized Schwarzschild spacetime the value of acceleration decreases asymptotically for small values of $r$and that enable the falling particle to move with steady velocity tell it hit the singularity. Equation

(38) means that the particle keep accelerating without limit, this gives infinity value of acceleration for small $r$. Thus it is convenient to use generalized acceleration and velocity for strong field when $\ll m$ , for large $r$Eqs (30) and (33) reduce to Eqs (36) and (38) of GR

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