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# Interaction Potential of Bounded Muon by Nucleus with Dirac Equation

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***Abstract:*** *The interaction potential between muon and nucleus, completely inside, completely outside and partially inside and outside was study. The interaction potential decease with increase in separation between muon and nucleus, and become minimum at $2.7fm$, muon completely inside the nucleus. The interaction potential goes decrease and become minimum at $2.7fm$ and then increase with distance separation between muon and nucleus, partially inside and outside of nucleus. When muon is completely outside the potential the nature of potential is like coulomb potential. The amplitude of Dirac wave function is maximum for $l = 2$ at $5fm$ and amplitude of Dirac wave function is same at certain potential for angular momentum $l = 0, 1, 2$.*
***Keywords:*** *Interaction potential, Nucleus, Muon, Dirac Wave Function, Amplitude, Angular momentum.*

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## I. Introduction

Anderson and Neddermeyer discovered the muon in 1937, muon is also known as fundamental building blocks of matter. Rasetti in 1941 demonstrated the life time of muon using plastic scintillator. The muon (μ) is an elementary particle with unit electric charge and ½ spin. Muon don't goes strongly interaction and it has partner called antimuon ($\mu^+$). The muon is quite unstable with decay mean lifetime about $\tau_\mu = 2.197\mu s$ [1]. The particle formation by muon after decay are,

$$\begin{aligned} \mu^+ &\rightarrow e^+ + v_e + \bar{v}_\mu \\ \mu^- &\rightarrow e^- + \bar{v}_e + v_\mu \end{aligned} \tag{1}$$

Muon is discover in 1930s and known as first second-generation standard-model particle of weak interaction, lepton number conservation during decay and play an important role in restricting theories of physics beyond standard model. The muon is capture by atomic nucleus and give the information of weak current by strong interaction by $\beta$ decay [2]. The muon is like electron ($q = \pm e$) with spin 1/2 and magnetic moment,

$$\mu_s = g_s \left(\frac{q}{2m}\right) s; \mu = (1 + a)\frac{q\hbar}{2m}; a = \frac{g_s - 2}{2} \tag{2}$$

Here $g_s$ is spin g-factor which is slightly greater than Dirac value 2, $\mu$ give the information of Dirac moment and Pauli moment.

The muon $g$ −factor (bounded by nucleus) is obtained by taking the ratio of precession frequency of a positive muon $f_\mu$ with proton $f_p$in presence of magnetic field is given as

$$g_\mu = g_p \frac{f_\mu}{f_p} \frac{m_\mu}{m_p} \frac{e_p}{e_\mu} \tag{3}$$

Here $g_p$ is g factor of proton, $m_\mu$ is mass of muon, $m_p$ is mass of proton, $e_p$ charge of proton and $e_e$ charge of electron. The Pauli momentum is obtained by changing polarization angle of muon in magnetic field for time t ($B_t$) is given [3] by

$$\frac{g-2}{2} = \frac{\theta(t) - \theta(0)}{\left(\frac{e}{m_\mu}\right) B_t} \tag{4}$$

The detail knowledge of various muonic-atomic and muonic-molecular processes is important in low-energy muon science. Studies of weak interaction, capture of muon by nucleus in muonic atoms and molecules. The spectrometry of muonic atoms enables to determine accurate electromagnetic structure of nuclei. The excitation of muonic atom radiate the energy, when muon exchanges between muonic hydrogen atoms and heavy element Strasser et al. (2005). The laser spectroscopy of muonic atoms are rise now a day to determine the precision on proton radius and testing of bounded state quantum electrodynamics Pohl et al. (2010) [4]. The interaction of muons with matter was studies by Hughes and Wu in 1975, the muon is heavier than positron and electron. When muon is stopped after interaction with matter it means muon is capture by nucleus and capture of muon may be completely inside the nucleus or completely outside or partially inside and outside. This different 3 type of capture of muon capturer are taken consideration in this work. There are two types of muon which get capture by atoms, they are positive and negative, the transition of muon capture by atoms or nucleus production the ration when transition of muon take place. The bounded or capture of nucleus is due to charge [5-6].

## II. Material And Methods

The Muonic atoms are used to study the nuclear properties because bounded or capture muon are very closure to nucleus and give high precision information about nucleons. The two nuclear properties was study by measuring the X-ray spectrum which is based on absolute nuclear charge radius and quadrupole moment of nucleus [7]. Since the capture of muon is take place by nucleus of an atom therefore here we divide the capturing region of nucleus. The region are divide into 3 type, muons are captured by nucleus inside it completely, muon are bounded by nucleus completely outside, and muon is capture and bounded partially by nucleus. The capture region of muon by nucleus are is divided into the three regions $r \leq R_N - R_\mu$ (muon completely inside the nucleus), $R_N - R_\mu < r < R_N + R_\mu$ (muon partially inside the nucleus) and $r > -R_N + R_\mu$ (muon completely outside the nucleus). Here $R_N$ is radius of nucleus, $R_\mu$ is radius of muonic charge and $r$ is distance between nuclear and muonic charge centers. The Interaction potential in genral form is assumed as

$$V(r) = -Ze^2 f_i(r) \quad i = 1,2,3 \tag{5}$$

**Muon Completely Inside the Nucleus**

From equation (5) the interaction $V(r)$ between nucleus and muon, when muon is inside the nucleus ($r < R_N - R_\mu$) is obtained as,

$$V(r) = -\frac{Ze^2}{R_N}\left(\frac{3}{2} - \frac{3}{10}R_\mu^2 R_N^{-2} - \frac{1}{2}r^2 R_N^{-2}\right) \tag{6}$$

**Muon Completely Outside the Nucleus**

From equation (5) the interaction $V(r)$ between nucleus and muon, when muon is completely outside the nucleus ($r \geq R_N + R_\mu$) is obtained as,

$$V(r) =— \frac{Ze^2}{r} \tag{7}$$

**Muon Partially Inside the Nucleus**

From equation (5) the interaction $V(r)$ between nucleus and muon, when muon is partially inside the nucleus ($R_N - R_\mu < r < R_N + R_\mu$) is obtained as,

$$V(r) = -Ze^2\left(\frac{d_{-1}}{r} + d_0 + d_1 r + d_2 r^2 + d_3 r^3 + d_5 r^5\right) \tag{8}$$

Where $d_{-1} = \frac{1}{2} + \frac{1}{32}\left(R_\mu R_N^{-1} + R_\mu^{-1} R_N\right)\left(R_\mu^2 R_N^{-2} - 10 + R_\mu^{-2} R_N^2\right)$, $d_0 = \frac{3}{4}\Big\{R_N^{-1}\left(1 - \frac{R_\mu^2 R_N^{-2}}{5}\right) + R_\mu^{-1}\Big(1 -$ *Rμ−2RN25, d1=932RN12Rμ−32−RN−32Rμ12, d2=−14RN−3+Rμ−3, d3=332RN−2+Rμ−2RNRμ−1,* $d_5 = -\frac{(R_N R_\mu)^{-3}}{160}$.

On rewriting the Dirac equation (1) of 1D first order differential equation for a radial potential from Sturniolo and Hiller [8] with $k = \left(j \pm \frac{1}{2}\right), l = j \mp \frac{1}{2}$ [9] is,

$$\begin{aligned} \frac{df(r)}{dr} &= \frac{k-1}{r} f(r) - \{E - mc^2 - V(r)\}\frac{g(r)}{\hbar c} \\ \frac{dg(r)}{dr} &= \frac{k+1}{r} g(r) + \{E + mc^2 - V(r)\}\frac{f(r)}{\hbar c} \end{aligned} \tag{9}$$

Defining Dirac function as $G(r) = rg(r)$, $F(r) = rf(r)$ and substituting in equation (9), after multiplying by r, computing the differentiation, canceling second order differential terms we get from equation (9).

$$\begin{aligned} \frac{dF(r)}{dr} &= \frac{k}{r}F(r) - \frac{1}{\hbar c}(E - mc^2 - V)G(r) \\ \frac{dG(r)}{dr} &= -\frac{k}{r}G(r) + \frac{1}{\hbar c}(E + mc^2 - V)F(r) \end{aligned} \tag{10}$$

Equation (10) are coupled radial equations with regular solutions inside the nucleus and the regular and the irregular solutions outside the nucleus. Also from equation (10), the Dirac radial equation is obtained as

$$\begin{aligned} \frac{d^2F(r)}{dr^2} + \frac{\left((E-V)^2 - m^2c^4\right)}{\hbar^2 c^2}F(r) - \frac{k(k-1)}{r^2}F(r) + \frac{dF(r)}{dr}\frac{dV}{dr}\frac{1}{(E-V-mc^2)} - \frac{k}{r}\frac{dV}{dr}F(r)\frac{1}{(E-V-mc^2)} &= 0 \\ \frac{d^2G(r)}{dr^2} + \frac{\left((E-V)^2 - m^2c^4\right)}{\hbar^2 c^2}G(r) - \frac{k(k+1)G(r)}{r^2} + \frac{dV}{dr}\frac{1}{(E-V+mc^2)}\frac{dG(r)}{dr} + \frac{dV}{dr}\frac{k}{r}G(r)\frac{1}{(E-V+mc^2)} &= 0 \end{aligned} \tag{11}$$

Supposing G(r)=$X_+(r)$ and F(r)=$X_-(r)$ and solving equation (11) yield

$$\left[\frac{d^2}{dr^2} + \left(\frac{dV}{dr}\frac{1}{(E-V\pm mc^2)}\right)\frac{d}{dr} \pm \frac{l}{r}\left(\frac{dV}{dr}\frac{1}{(E-V\pm mc^2)}\right) - \frac{l(l\pm 1)}{r^2} + \frac{(E-V)^2 - m^2c^4}{\hbar^2 c^2}\right]X_\pm(r) = 0 \tag{12}$$

Substituting $X_\pm(r) = r\psi_\pm(r)$, here $\psi_+(r) = g(r)$ and $\psi_-(r) = f(r)$ in equation (12) and solving we get,

$$\left[\frac{d^2}{dr^2} + \left(\frac{2}{r} + \frac{dV}{dr}\frac{1}{(E-V\pm mc^2)}\right)\frac{d\psi}{dr} + \frac{(1\pm k)}{r}\left(\frac{dV}{dr}\frac{1}{(E-V\pm mc^2)}\right) - \frac{k(k\pm 1)}{r^2} + \frac{(E-V)^2 - m^2c^4}{\hbar^2 c^2}\right]\psi_\pm(r) = 0 \tag{13}$$

Substituting the value of $V(r) = -\frac{Ze^2}{R_N}\left(\frac{3}{2} - \frac{3}{10}R_\mu^2 R_N^{-2} - \frac{1}{2}r^2 R_N^{-2}\right)$ in equation (13) and solving we get,

$$\left[\frac{d^2}{dr^2} + \left(\frac{2}{r} + \frac{Ze^2 r}{R_N^3}\frac{1}{(E-V\pm mc^2)}\right)\frac{d\psi}{dr} + \frac{Ze^2(1\pm l)}{R_N^3}\left(\frac{1}{(E-V\pm mc^2)}\right) - \frac{l(l\pm 1)}{r^2} + \frac{(E-V)^2 - m^2c^4}{\hbar^2 c^2}\right]\psi_\pm(r) = 0 \tag{14}$$

Omitting the $R_N$ (since nucleus is consider as point charge and muon is inside the nucleus at a distance r.

$$\frac{d^2\psi\pm}{dr^2}+\frac{2}{r}\frac{d\psi\pm}{dr}+\left[-\beta^2-\frac{l(l\pm1)}{r^2}\right]\psi_\pm=0 \tag{15}$$

Where $\frac{(E-V)^2-m^2c^4}{\hbar^2c^2}=-\beta^2$, let us supposed $\rho=2\beta r \Rightarrow \frac{d\rho}{dr}=2\beta, \frac{d\psi}{dr}=\frac{d\psi}{d\rho}\frac{d\rho}{dr}$ then from equation (15) we have,

$$\frac{d^2\psi\pm(\rho)}{d\rho^2}+\frac{2}{\rho}\frac{d\psi\pm(\rho)}{d\rho}+\left[-\frac{1}{4}-\frac{l(l\pm1)}{\rho^2}\right]\psi_\pm(\rho)=0 \tag{16}$$

When $\rho\to\infty$ the solution of equation (16) is $\psi_\pm(\rho)=e^{\pm\frac{\rho}{2}}$ this solution is not valid for $e^{+\frac{\rho}{2}}$ when $\rho\to\infty$ but valid for larger $e^{-\frac{\rho}{2}}$ for $\rho\to\infty$. Therefore we select $e^{-\frac{\rho}{2}}$ at $\rho\to\infty$, it is quite reasonable to think that the exact solution may also contain some pre-exponential part to attain validity at all value of $\rho$. Therefore after incorporating some $\rho$-dependent unknown function $F(\rho)$ then the solution

$$\psi_-(\rho)=F(\rho)e^{-\frac{\rho}{2}} \tag{17}$$

On substituting equation (17) in equation (16) we get,

$$\frac{d^2F(\rho)}{d\rho^2}+\left(\frac{2}{\rho}-1\right)\frac{dF(\rho)}{d\rho}+\left[-\frac{1}{\rho}-\frac{l(l\pm1)}{\rho^2}\right]F(\rho)=0 \tag{18}$$

Hence the problem has been reduced to the determine of the solution of $F(\rho)$ which is assumed as

$$F(\rho)=\rho^sG(\rho) \tag{19}$$

Where $G(\rho)$ represent as power series of $\rho$ as

$$G(\rho)=a_0+a_1\rho+a_2\rho^2+a_3\rho^3+\cdots\ldots.=\sum_{m=0}^{m=\infty}a_m\rho^m\ ,for\ a_0\neq0 \tag{20}$$

On differentiation equation (19) w.r.t to $\rho$ we get, $F'(\rho)=s\rho^{s-1}G+s^sG, F''(\rho)=s(s-1)\rho^{s-2}+2s\rho^{s-1}G'+\rho^sG''$ and substituting value of $F'(\rho), F(\rho), F''(\rho)$ in (18) and multiply by $4\rho^2$ and dividing with $\rho^s$ on solving we get

$$[4s(s-1)+8s-4s\rho-4\rho-4l(l+1)]G+[8s\rho+8\rho-4\rho^2]G'+4\rho^2G''=0 \tag{21}$$

If $\rho=0$, then the $G(\rho)=a_0$ then equation (21) become

$$[4s(s-1)+8s-4l(l+1)]G=0 \tag{22}$$

On solving we get

$$s(s+1)=l(l+1)=0 \tag{23}$$

This implies that $s=l$ or $s=-(l+1)$ if we consider s$=-(l+1)$ then $F(\rho)$ has infinite solution at ($\rho=0$) and not acceptable. Therefoere $s=l$ is acceptable and equation (21) become

$$[4l(l-1)+8l-4l\rho-4\rho-4l(l+1)]G+[8l\rho+8\rho-4\rho^2]G'+4\rho^2G''=0 \tag{24}$$

On solving we get,

$$[-4l\rho-4\rho]G+[8l\rho+8\rho-4\rho^2]G'+4\rho^2G''=0 \tag{25}$$

On dividing by $4\rho$ we get,

$$[-l\rho-\rho]G+[2l\rho+2\rho-\rho^2]G'+\rho^2G''=0 \tag{26}$$

On substituting value $G(\rho), G'(\rho)\ and\ G''(\rho)$ from above in equation (26) we get,

$$[-l-1]\sum_{m=0}^{m=\infty}a_m\rho^m+[2l+2-\rho]\sum_{m=0}^{m=\infty}a_mm\rho^{m-1}+\rho\sum_{m=0}^{m=\infty}a_mm(m-1)\rho^{m-2}=0 \tag{27}$$

Equation (27) hold truly if the coefficient of individual power of $\rho$ become zero. So, simplifying equation (27) for two summation term ($a_m$ and $a_{m+1}$) we have

$$[-l-1](a_m\rho^m+a_{m+1}\rho^{m+1})+[2l+2-\rho](a_mm\rho^{m-1}+a_{m+1}(m+1)\rho^m)+\rho(a_mm(m-1)\rho^{m-2}+a_{m+1}(m+1)m\rho^{m-1})=0$$
(28)

On solving and putting a coefficient of $\rho^k$ equal to zero, we get

$$a_{m+1}=\frac{l+1+m}{(m+1)(2l+m+2)}a_m \qquad for\ l=j\mp\frac{1}{2} \tag{29}$$

This expression allows one to determine the coefficient ($a_{m+1}$) in term of $a_m$ which is arbitrary. Now, since series $G(\rho)$ consist of the infinite number of term, the function $F(\rho)$ become infinite at a very large value of m i.e. infinite. Consequently the function $\psi_-(\rho)$ will also become infinite if the number of terms is not limited to a finite value. Therefore, we must break off the series to a finite number of term which is possible only if the numerator become zero i.e.

$$l+1+m=0 \tag{30}$$

Therefore the series of solution of Dirac equation is obtained from equation (17), (19) and (29) for $m=0,1,2,3$ and $s=l$, as

$$\psi_-(\rho)=\rho^le^{-\frac{\rho}{2}}a_0\ \left(1+\frac{\rho}{2}+\frac{(l+2)}{4(2l+3)}\rho^2+\frac{l+3}{24(2l+3)}\rho^3+\frac{l+4}{96(2l+5)}\frac{l+3}{24(2l+3)}\rho^4\right) \tag{31}$$

This is the solution of Dirac equation when the muon is completely inside the nucleus. In similar manner one can obtained the solution of Dirac equation when muon is completely outside the nucleus and partially inside the nucleus.

## III. Results and Discussion

The interaction potential decrease with increasing the separation of nucleus and muon up to certain distance 0 to 1.3fm, when muon is inside the nucleus. The decreasing in interaction potential between the nucleus and muon is due to repulsion between them as explain by muon theory (Yukawa potential). The interaction potential of muon, inside the nucleus are shown in Fig.1. Fig.1 is used to study the nature of interaction potential between muon and nucleus of hydrogen atom, the calculation is based on atomic unit. The interaction potential was found maximum at center as shown in Fig.1.

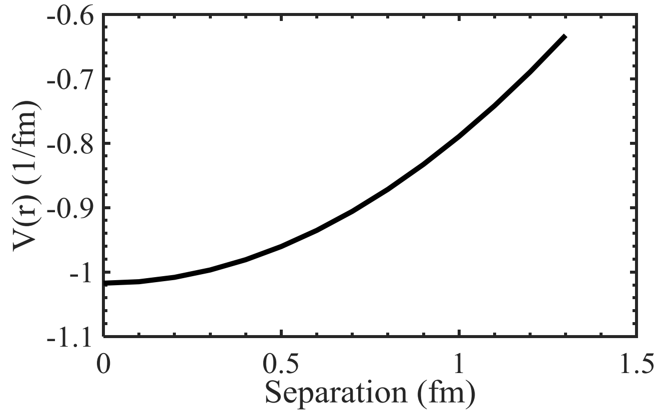


Fig.1. Interaction potential between Muon and nucleus, completely inside the nucleus.

The interaction potential between muon and nucleus is high 2.7fm and with increasing the separation the interaction potential goes decreases, muon is completely outside the nucleus. The nature of interaction potential in such case is like coulomb potential, at a larger distance the interaction potential is infinity. This implies that the interaction potential is valid only below 100fm or less. The nature of interaction potential between muon and nucleus are shown in Fig.2.

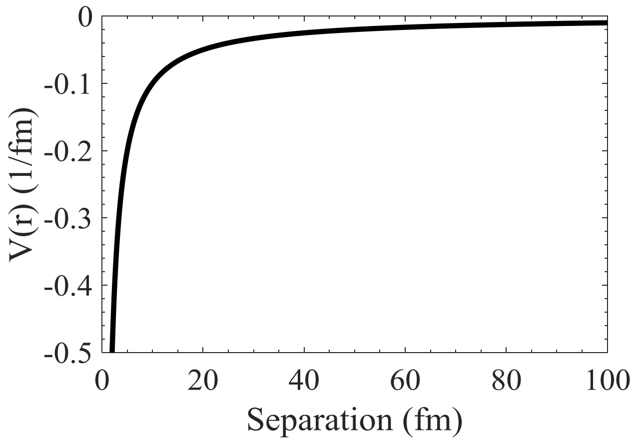


Fig.2. Interaction between Muon and nucleus, completely outside the nucleus.

The interaction potential between muon and nucleus decrease with increasing the distance separation between them, muon is partially inside and outside the nucleus. The interaction potential is minimum at point 2.7fm (4.86fm$^{-1}$), muon partially inside the nucleus and the interaction potential increase with distance increase up to 5fm (Fig.3a), muon is partially outside the nucleus. The nature of interaction potential when muon is partially in and outside the nucleus is shown in Fig.3a. The interaction decrease with increasing the separation between the muon and nucleus for partially inside is due to repulsion between muon and nucleus and the interaction potential increase for partially outside is due to attraction between muon and nucleus.

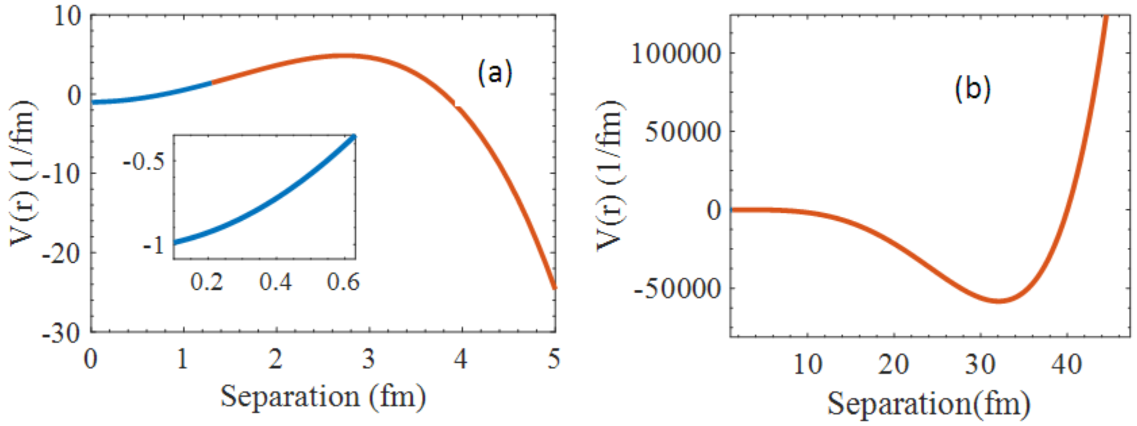


Fig.3. Interaction between Muon and nucleus, partially in and outside the nucleus.

Also from Fig.3b it is clear that the interaction potential increase with increasing the separation of muon and nucleus, partially in and outside the nucleus. The interaction potential is maximum at 3.2fm (-23320.177$fm^{-1}$) and then goes sharply decrease as shown in Fig.3b.
The amplitude for the interaction potential, muon is inside the nucleus is shown in Fig.4, Fig.4a is based on interaction potential $V(r) = -0.6fm^{-1}$, energy $E$ =0.1MeV, $a_0 = 1$ for hydrogen atom for angular momentum $0, 1, 2$. The amplitude is maximum for $l = 2$, about $5fm$ distance separation between muon and nucleus, visualized in Fig.4a.

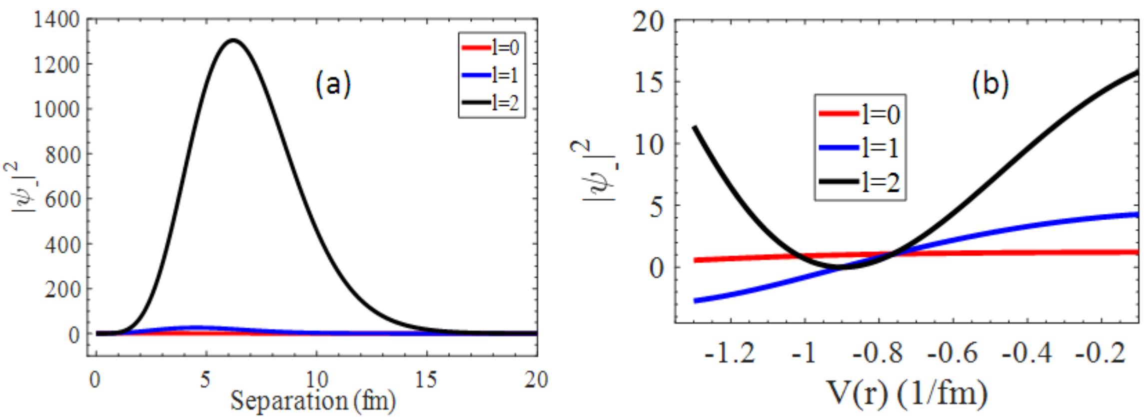


Fig.4. Amplitude of Dirac wave function, muon completely inside the nucleus

Fig.4b shows that amplitude of Dirac wave function decrease with decreasing interaction potential and at a certain value $-0.9fm^{-1}$ is lowest and then increase with decreasing the interaction potential, for $l = 2$. The amplitude increase with decrease interaction potential for $l = 1$ and constant for $l = 0$. The amplitude of Dirac wave function is same for $l = 0$ and $l = 1$ at potential $-0.7615526fm^{-1}$ is 1.07609, $l = 0$ and $l = 2$ at potential $-0.762928fm^{-1}$ is 0.0755, and for $l = 1$ and $l = 2$ at $-0.764296fm^{-1}$ at 1.0555 as shown in Fig.4b.

## IV. Conclusion

The interaction potential, when muon is inside the nucleus is decrease with increase the distance verified muon theory, muon outside the nucleus verified the coulomb potential and muon partial inside shows that interaction goes decrease to a minimum value and increase up to maximum value with distance separation between them. The amplitude of Dirac function for $l = 2$ is maximum when studies with separation of distance between muon and nucleus inside the nucleus. The amplitude of Dirac wave function is equal for $l = 0$ and $l = 1, l = 0$ and $l = 2, l = 1$ and $l = 2$ when observed with interaction potential.

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