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# The Torsion Tensor Field $\Omega^i_{jk}$ As a Parallel Transported Field

Alcantara-Montes Samuel and Velazquez-Arcos Juan Manuel
*Basic Sciences/Metropolitan Autonomous University, Mexico*

***Abstract:*** *The condition $\nabla_l\Omega^i_{jk}$=0 in Einstein identity allows us to write the Riemann curvature tensor as a product of the torsion tensor and then in terms of the electromagnetic field tensor $f_{ij}$ .*
***Key Words:*** *Torsion tensor, Curvature tensor, Electromagnetic field tensor.*

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## I. Introduction

In a 4-dimensional continuum $L^i_{jk}(x) = \Gamma^i_{jk}(x) + \Omega^i_{jk}(x)$ [1] is the asymmetrical affine connection of the space, where $\Gamma^i_{jk}$ are the Christoffel symbols and $\Omega^i_{jk}$ is the torsion tensor field. The field $L^i_{jk}(x)$ allow us to construct the covariant derivative of a parallel vector field $A^i$ and $A_i$ in the following way:

$$\nabla_l A^i = \frac{\partial A^i}{\partial x^l} + L^i_{hl}A^h = 0 \quad (1.1)$$

$$\nabla_l A_i = \frac{\partial A_i}{\partial x^l} - L^h_{il}A_h = 0 \quad (1.2)$$

Indexes run from one to four .The condition of integrability of (1.1) is:

$$A^j L^i_{jkl} = 0 \quad (1.3)$$

Or simply as:

$$L^i_{jkl} = 0 \quad (1.4)$$

Where:

$$L^i_{jkl} = \frac{\partial L^i_{jl}}{\partial x^k} - \frac{\partial L^i_{jk}}{\partial x^l} + L^i_{hk}L^h_{jl} - L^i_{hl}L^h_{jk} \quad (1.5)$$

But: $L^i_{jk} = \Gamma^i_{jk} + \Omega^i_{jk}$ then:

$$L^i_{jkl} = R^i_{jkl} + \Omega^i_{jkl} \quad (1.6)$$

Where:

$$R^i_{jkl} = \frac{\partial \Gamma^i_{jl}}{\partial x^k} - \frac{\partial \Gamma^i_{jk}}{\partial x^l} + \Gamma^i_{hk}\Gamma^h_{jl} - \Gamma^i_{hl}\Gamma^h_{jk} \quad (1.7)$$

$$\Omega^i_{jkl} = \Omega^i{}_{jl|k} - \Omega^i_{jk|l} + \Omega^i_{hl}\Omega^h_{jk} - \Omega^i_{hk}\Omega^h_{jl} - 2\Omega^i_{jh}\Omega^h_{kl} \quad (1.8)$$

A solidus followed by an index indicates covariant differentiation with respect to the field $L^i_{jk}$ .

$R^i_{jkl}$ as defined by (1.7) are the components of the Riemann curvature tensor of the symmetric connection coefficient $\Gamma^i_{jk}$ , well known as the Christoffel symbols, they satisfies the identities [2] :

$$R^i_{jkl} + R^i_{jlk} = 0 \quad (1.9)$$

$$R^i_{jkl} + R^i_{klj} + R^i_{ljk} = 0 \quad (1.10)$$

From parallelism condition:

$$L^i_{jkl} = 0 \quad (1.11)$$

We have: [3]

$$R^i_{jkl} + \Omega^i_{jkl} = 0 \quad (1.12)$$

And from (1.10):

$$\Omega^i_{jkl} + \Omega^i_{klj} + \Omega^i_{ljk} = 0 \quad (1.13)$$

Permutation of the indexes j k l in (1.8) and then add, according to (1.13), gives:

$$\Omega^i_{jl|k} + \Omega^i_{lk|j} + \Omega^i_{kj|l} + 2\left(\Omega^h_{jl}\Omega^i_{kh} + \Omega^h_{lk}\Omega^i_{jh} + \Omega^h_{kj}\Omega^i_{lh}\right) = 0 \quad (1.14)$$

This identity discovered by Einstein in 1929, can it be obtained from Ricci and Jacobi identities [4].
In section II we consider the condition $\nabla_l \Omega^i_{jk} = 0$ . In section III we find Maxwell equations.

## II. The zero covariant derivative of $\Omega^i_{jk}$

If the covariant differentiation with respect to $L^i_{jk}$ for $\Omega^i_{jk}$ is equal to zero [5], we have from (1.13) that:

$\Omega^h_{jl}\Omega i_{kh} + \Omega^h_{lk}\Omega^i_{jh} + \Omega^h_{kj}\Omega^i_{lh} = 0$ (2.1)
Then from (1.8) and (2.1):
$\Omega^i_{jkl} = -\Omega^h_{kl}\Omega^i_{jh}$ (2.2)
And from (1.12), [5]
$R^i_{jkl} = \Omega^i_{jh}\Omega^h_{kl}$ (2.3)
This formula is very interesting .It has to do with a transitive group.
Now from the definition:
$\Omega^i_{jk} = \frac{e}{mc^2}u^i f_{jk}$ (2.4)
$R^i_{jkl} = \left(\frac{e}{mc^2}\right)^2 u^h u^i f_{jh} f_{kl}$ (2.5)

A very attractive and interesting result. The Riemann curvature tensor is generated by the electromagnetic field tensor $f_{ij}$. This formula is a consequence of Einstein's identity (1.14). m is the particle mass, e is the electrical charge and c is light velocity .

## III. Maxwell equations

From (2.4) the connection between the torsion tensor $\Omega^i_{jk}$ and the electromagnetic field tensor $f_{ij}$ , we have:

$\nabla_l \Omega^i_{jk} = \frac{e}{mc^2}\nabla_l\left(u^i f_{jk}\right) = 0$ (3.1)

But $\nabla_l u^i = 0$ . Then:

$\nabla_l f_{jk} = 0$ (3.2)

But:

$\nabla_l f_{jk} = \frac{\partial f_{jk}}{\partial x^l} - L^h_{jh} f_{hk} - L^h_{kl} f_{jh} = 0$ (3.3)

Therefore:

$\frac{\partial f_{jk}}{\partial x^l} = L^h_{jl} f_{kh} + L^h_{kl} f_{jh}$ (3.4)

$\frac{\partial f_{jk}}{\partial x^1} = \Gamma^h_{jl} f_{kh} + \Gamma^h_{kl} f_{jh} + \Omega^h_{jl} f_{hk} + \Omega^h_{kl} f_{jh}$ (3.5)

But from (1.12) and (2.4) we have:

$\Omega^h_{jl} f_{hk} + \Omega^h_{kl} f_{jh} = \frac{e}{mc^2} u_i R^i_{ljk}$ (3.6)

Then:

$\frac{\partial f_{jk}}{\partial x^l} = \Gamma^h_{jl} f_{hk} + \Gamma^h_{kl} f_{jh} + \frac{e}{mc^2} u_i R^i_{ljk}$ (3.7)

These equations show a relation between the gradient of the electromagnetic field tensor and the gradient of the gravitational field though the components of the Riemann curvature tensor.

If in (3.7) we put k=l, we have:

$$\frac{\partial f_{il}}{\partial x^l} = \Gamma^h_{jl} f_{hl} + \Gamma^h_{ll} f_{jh} + \frac{e}{mc^2} u_i R^i_{ljl} \quad (3.8)$$

The divergences of the electromagnetic field tensor is no more equal to zero.

Now, the four Maxwell equations follows from (3.7) and (3.8) .

We know that:

$$R^i_{jkl} + R^i_{klj} + R^i_{ljk} = 0 \quad (3.9)$$

Now, if in (3, 7) we make a permutation of the indexes j k l and then add, we have:

$$\frac{\partial f_{jk}}{\partial x^l} + \frac{\partial f_{kl}}{\partial x^j} + \frac{\partial f_{lj}}{\partial x^k} = 0 \quad (3.10)$$

(3.10) are the first two Maxwell equations:

$$\nabla \cdot \vec{H} = 0 \quad (3.11)$$

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{H}}{\partial t} \quad (3.12)$$

From (3.8), if $\Gamma^i_{jk} = 0$, we have:

$$\frac{\partial f_{il}}{\partial x^l} = 0 \quad (3.13)$$

Which we easily recognize as the last two equations:

$$\nabla \cdot \vec{E} = 0 \quad (3.14)$$

$$\nabla \times \vec{H} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} \quad (3.15)$$

## IV. Conclusions

Equation (1.12) says: Riemann curvature tensor cannot exist without the torsion tensor

The 4-dimensional connection $L^i_{jk}(x)= \Gamma^i_{jk}(x) + \Omega^i_{jk}(x)$ can be written explicitly as a sum of the gravitational field $\Gamma^i_{jk}$ and the electromagnetic field $\Omega^i_{jk} = \frac{e}{mc^2} u^i f_{jk}$ .

Einstein identity admits two treatments. First, we demand that the tensor field $\Omega^i_{jk}$ satisfies the condition: $\Omega^h_{jl}\Omega^i_{kh} + \Omega^h_{lk}\Omega^i_{jh} + \Omega^h_{kj}\Omega^i_{lh} = 0$ which implies $\Omega^i_{jl|k} + \Omega^i_{lk|j} + \Omega^i_{kj|l} = 0$ .The consequences of that request were discussed in a previous article [4] .Second we require that the covariant derivative of the tensor field $\Omega^i_{jk}$ is equal to zero, which implies that the product of the tensor field $\Omega^i_{jk}$ is equal to zero. The consequences of this request are presented in this article .Both request give almost the same result.

Equation (2.5) is unexpected, it demonstrates the dependence of the Riemann curvature tensor on the electromagnetic field tensor. Its interpretation is obvious**: Gravitation is generated by the electromagnetic field.**

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