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Parallel Transport in 4-dimensional Continuum

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Abstract:*We consider a 4-dimensional continuum with asymmetrical affine connection $L^i_{jk}(x)$, a field which defines infinitesimal vector shifts according to the relation $dA^i = -L^i_{jk} A^j\, dx^k$, or in a more simple way as the covariant derivative of the contravariant field A^i, i. e. $\nabla_j A^i = 0$. This equation defines a parallel vector field A^i, and at the same time, the equation of motion of a particle.*

Key Words:*Asymmetrical field, Scalar field, Tensor field, Transporter.*

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I. Introduction

As is well known, in a system of coordinate X, a contravariant parallel vector field A^i and a covariant parallel vector field A_i satisfies the systems of differentials equations:

$$\frac{\partial A^i}{\partial x^j} + \Gamma^i_{kj} A^k = 0 \ (1.1)$$

$$\frac{\partial A_i}{\partial x^j} - \Gamma^k_{ij} A_k = 0 \ (1.2)$$

Where $\Gamma^i_{jk}(x)$ are the Christoffel symbols of the second kind formed with respect to the fundamental tensor $g_{ij\,(x)}$ of the space and are the transporters of the contravariant and covariant parallel vectors A^i and A_i respectively .

In two system of coordinates X and Y Christoffel symbols are connected by the relation:

$$\Gamma^\gamma_{ij}(y)\frac{\partial x^m}{\partial y^\gamma} = \frac{\partial x^\alpha}{\partial y^i}\frac{\partial x^\beta}{\partial y^j}\Gamma^m_{\alpha\beta}(X) + \frac{\partial^2 x^m}{\partial y^i \partial y^j} \ (1.3)$$

The covariant derivatives (1.1) and (1.2) can be generalized if we use L^i_{jk} as a transporter which satisfies the systems of differentials equations:

$$L^\gamma_{\alpha\beta}(y)\frac{\partial x^i}{\partial y^\gamma} = \frac{\partial x^j}{\partial y\alpha}\frac{\partial x^k}{\partial y^\beta} L^i_{jk}(X) + \frac{\partial^2 x^i}{\partial y^\alpha \partial y^\beta}(1.4)$$

[1],[2],[3]. i ,j ,k,α, β, γ=1,2,3,4.

If we use ∇ or a bar (|) to represent the covariant derivative, then:

$$\nabla_j A^i = \frac{\partial A^i}{\partial x^j} + L^i_{kj} A^k \ (1.5)$$

$$\nabla_j A_i = \frac{\partial A_i}{\partial x^j} - L^k_{ij} A_k \ (1.6)$$

Now let us put:

$$\Gamma^i_{jk} = \frac{1}{2}\left(L^i_{jk} + L^i_{kj}\right) (1.7)$$

$$\Omega^i_{jk} = \frac{1}{2}\left(L^i_{jk} - L^i_{kj}\right)(1.8)$$

And substitute $L^i_{jk} = \Gamma^i_{jk} + \Omega i_{jk}$ in (1.4), it follows that Γ^i_{jk} satisfies (1.3) and Ω^i_{jk} behaves as a tensor, known as the torsion tensor.
Now, a field of contravariant and covariant parallel vectors A^i and A_i must satisfies the systems of differentials equations:

$\frac{\partial A^i}{\partial x^j} + L^i_{kj} A^k = 0$ (1.9)

$\frac{\partial A_i}{\partial x^j} - L^k_{ij} A_k = 0$ (1.10)

The condition of integrability of (1.9) is given by:

$A^j L^i_{jkl} = 0$ (1.11)

Or:

$L^i_{jkl} = 0$ (1.12)

Where:

$L^i_{jkl} = \frac{\partial L^i_{jl}}{\partial x^k} - \frac{\partial L^i_{jk}}{\partial x^l} + L^i_{hl} L^h_{jk} - L^i_{hk} L^h_{jl}$ (1.13)

Now, the parallelism condition requires that, [4], [5], [6]:

$L^i_{jkl} = 0$ (1.14)

From Ricci generalized identities:

$[\nabla_j, \nabla_k] A^i = -L^i_{hkj} A^h - 2\Omega^h_{kj} \nabla_h A^i$ (1.14)

$[\nabla_j, \nabla_k] A_i = L^h_{ikj} A_h - 2\Omega^h_{kj} \nabla_h A_i$ (1.15)

And considering A^i and A_i a contravariant and a covariant parallel vector fields, we have the commutator of the covariant derivatives as:

$[\nabla_j, \nabla_k] = 2\Omega^h_{jk} \nabla_h$ (1.16)

And it is clear that Jacobi identity must be satisfied:

$\left[\nabla_i, [\nabla_j, \nabla_k]\right] + \left[\nabla_j, [\nabla_k, \nabla_i]\right] + \left[\nabla_k, [\nabla_i, \nabla_j]\right] = 0$ (1.17)

From (1.16) and (1.17) we have:

$$\Omega^h_{ij|k} + \Omega^h_{jk_{|i}} + \Omega_{ki|j} + 2\left(\Omega^l_{ij}\Omega^h_{kl} + \Omega^l_{jk}\Omega^h_{il} + \Omega^l_{ki}\Omega^h_{jl}\right) = 0 \qquad (1.18)$$

An identity discovered by Einstein I 1929. [7],[8].
If in (1.18) we demand that the tensor field Ω^i_{jk} satisfies the condition:

$\Omega^l_{ij}\Omega^h_{kl} + \Omega^l_{jk}\Omega^h_{il} + \Omega^l_{ki}\Omega^h_{jl} = 0$(1.19)

Then:

$\Omega^h_{ij|k} + \Omega^h_{jk_{|i}} + \Omega_{ki|j} = 0$ (1.20)

Equation (1.20) will be very useful in constructing Maxwell's equations

Let us substitute $L^i_{jk} = \Gamma^i_{jk} + \Omega^i_{jk}$ in (1.13), we have:

$L^i_{jkl} = R^i_{jkl} + S^i_{jkl}$ (1.21)

Where:
$R^i_{jkl} = \frac{\partial \Gamma^i_{jl}}{\partial x^k} - \frac{\partial \Gamma^i_{jk}}{\partial x^l} + \Gamma^i_{hk}\Gamma^h_{jl} - \Gamma^i_{hl}\Gamma^h_{jk}$
(1.22)

$s^i_{jkl} = \Omega^i_{jl|k} - \Omega^i_{jk|l} + \Omega^i_{l}\Omega^h_{jk} - \Omega^i_{hk}\Omega^h_{jl} - 2\Omega^i_{jh}\Omega^h_{kl}$
(1.23)

And from (1.19) and (1.20) :

$s^i_{jkl} = \Omega^i_{kl|j} - \Omega^h_{kl}\Omega^i_{jh}$(1.24)

The tensor R^i_{jkl} must be recognized as the Riemann curvature tensor of the space , and the tensor s^i_{jkl} is the so called `` torsional tensor `` and is intimately connected whit the electromagnetic field.
In section II we discuss the tensor field Ω^i_{jk} as a transporter, and connect the tensor field with the electromagnetic field. In section III we generalize Lorentz equation of motion and find the connection between electromagnetic field and gravitational field.

II. The tensor field $\boldsymbol{\Omega^i_{jk}}$ as a transporter

The asymmetric transporter L^i_{jk} is composed by Γ^i_{jk} and Ω^i_{jk} as

$L^i_{jk} = \Gamma^i_{jk} + \Omega^i_{jk}$ (2.1)

That is like a sum of a scalar field Γ^i_{jk} and a tensor field Ω^i_{jk} , which remind us the structure of Hamilton quaternions.
Now, if we have a space where $\Gamma^i_{jk} = 0$ then:

$L^i_{jk} = \Omega^i_{jk}$ (2.2)

And the covariant derivative is:

$\nabla_j A^i = \frac{\partial A^i}{\partial x^j} + \Omega^i_{kj} A^k$ (2.3)

$\nabla_j A_i = \frac{\partial A_i}{\partial X^j} - \Omega^k_{ij} A_k$(2.4)

And consequently if A^i and A_i are contravariant and covariant parallel vector fields, a natural equation for these vectors are:

$\frac{\partial A^i}{\partial x^j} + \Omega^i_{kj} A^k = 0$ (2.5)

$\frac{\partial A_i}{\partial x^j} - \Omega^k_{ij} A_k = 0$ (2.6)

(2.5) and (2.6) are very interesting in a flat pseudo Euclidean space.

III. The electromagnetic tensor

The covariant equation (2.6) is a very suggestive one because if A_i is a parallel covariant vector velocity field u_i, then:

$$\frac{\partial u_i}{\partial x^j} - \Omega_{ij}^k u_k = 0 \quad (3.1)$$

And if we have a curve x = x(s) and u = u(x),then:

$$\frac{du_i}{ds} = \frac{\partial u_i}{\partial x^j}\frac{dx^j}{ds} = \frac{\partial u_i}{\partial x^j} u^j \quad (3.2)$$

And from (3.1):

$$\frac{du_i}{ds} = \Omega_{ij}^k u_k u^j \quad (3.3)$$

Now if we define the tensor field Ω_{ij}^k as:

$$u_k \Omega_{ij}^k = \frac{e}{mc^2} f_{ij} \quad (3.4)$$

Where f_{ij} is the well-known electromagnetic tensor field .

We have:

$$mc\frac{du_i}{ds} = \frac{e}{c} f_{ij} u^j \quad (3.5)$$

Which we can recognize as the famous H. Lorentz equation of motion of a charge in the electromagnetic field in a 4-dimensional form [9], m is the particle mass, e is the electrical charge and c is light velocity .

Equation (3.1) and definition (3.3) permit us to understand that Ω_{ij}^k is the parallel transporter in a 4-dimensional space, a space of zero curvature.

Parallelism implies $L_{jkl}^i = 0$ and if $\Gamma_{jk}^i = 0$ we have $s_{jkl}^i = 0$.

If:

$$S_{jkl}^i = 0 \quad (3.6)$$

Using (1.24) we have:

$$\Omega_{kl|j}^i = \Omega_{kl}^h \Omega_{jh}^i \quad (3.7)$$

But:

$$\Omega_{lk|j}^i = \frac{\partial \Omega_{lk}^i}{\partial x^j} + \Omega_{hj}^i \Omega_{lk}^h - \Omega_{lj}^h \Omega_{hk}^i - \Omega_{kj}^h \Omega_{lh}^i \quad (3.8)$$

If we put k= j in (3.8), we have

$$\Omega^i_{lj\,|j} = \frac{\partial \Omega^i_{lj}}{\partial x^j} - \Omega^i_{hj}\Omega^h_{lj} - \Omega^h_{lj}\Omega^i_{hj} \quad (3.9)$$

And (3.7) gives:

$$\Omega^i_{jl\,|j} = \Omega^h_{jl}\Omega^i_{jh} \quad (3.10)$$

Comparing (3.9) and (3.10) we have:

$$\frac{\partial \Omega^i_{lj}}{\partial x^j} + \Omega^i_{hj}\Omega^h_{lj} = 0 \quad (3.11)$$

And multiplication by u_i and using (3.1) gives:

$$u_i \frac{\partial \Omega^i_{lj}}{\partial x^j} + u_i \Omega^i_{hj}\Omega^h_{lj} = 0 \quad (3.12)$$

Or:

$$\frac{\partial}{\partial x^j}\left(u_i \Omega^i_{lj}\right) = 0 \quad (3.13)$$

And from (3.3):

$$\frac{\partial f_{lj}}{\partial x^j} = 0 \quad (3.14)$$

This can be recognized easily, as two of the Maxwell equations:

$$\nabla . \vec{E} = 0 \quad (3.15)$$

$$\nabla x \vec{H} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} \quad (3.16)$$

The other two Maxwell equations follow directly from (1.20) and using (1.19) i.e.:

$$\frac{\partial f_{ij}}{\partial x^k} + \frac{\partial f_{jk}}{\partial x^l} + \frac{\partial f_{ki}}{\partial x^j} = 0 \quad (3.17)$$

Or:

$$\nabla . \vec{H} = 0 \quad (3.18)$$

$$\nabla x \vec{E} = -\frac{1}{C}\frac{\partial \vec{H}}{\partial t} \quad (3.19)$$

After the interpretation of the tensor field Ω^i_{jk} as the parallel transporter in Minkowski space and intimately connected with the electromagnetic field f_{ij}, we can return to the covariant equation (1.10) i.e. :

$$\frac{\partial A_i}{\partial x^j} - L^k_{ij} A_k = 0 \quad (3.20)$$

Or:

$$\frac{\partial A_i}{\partial x^j} - \Gamma^k_{ij} A_k - \Omega^k_{ij} A_k = 0 \quad (3.21)$$

If we use (3.21) for a parallel velocity vector field u_i , we have:

$\frac{\partial u_i}{\partial x^j} - \Gamma_{ij}^k u_k - \Omega_{ij}^k u_k = 0$ (3.22)

And for a curve x = x(s) and u = u (x) and definition (3.3) we have:

$\frac{du_i}{ds} = \Gamma_{ij}^k u_k u^j + \frac{e}{mc^2} f_{ij} u^j$ (3.21)

Or:

$\frac{e}{c} f_{ij} u^j = mc\left(\frac{du_i}{ds} - \Gamma_{ij}^k u_k u^j\right)$(3.22)

As a generalization of H. Lorentz equation on motion of a charged particle, but now including Gravitational field Γ_{jk}^i [9] , [10] ,[11],[12].

It is very interesting to see how Maxwell equation behaves when the gravitational field is present.

From parallelism condition $L_{jkl}^i = 0$ we have:

$R_{jkl}^i + S_{jkl}^i = 0$ (3.23)

And from (1.24):

$R_{jkl}^i + \Omega_{kl|j}^i - \Omega_{kl}^h \Omega_{jh}^i = 0$ (3.24)

Multiplication by u_i gives:

$f_{kl|j} - \Omega_{kl}^h f_{jh} + \frac{mc^2}{e} u_i R_{jkl}^i = 0$ (3.25)

And using the covariant derivative of $f_{kl|j}$, wehave:

$\frac{\partial f_{kl}}{\partial x^j} - \Gamma_{kj}^h f_{hl} - \Omega_{kj}^h f_{hl} - \Gamma_{lj}^h f_{kh} - \Omega_{kl}^h f_{jh} + \frac{mC^2}{e} u_i R_{jkl}^i = 0$ (3.26)

These equations show a relation between the gradient of the electromagnetic field tensor an the gradient of the gravitational field through the components of the Riemann tensor.

Now, if in (3.26) we put l = j we have:

$\frac{\partial f_{kj}}{\partial x^j} = \Gamma_{kj}^h f_{hj} + \Gamma_{jj}^h f_{kh} + \frac{mc^2}{e} u_i R_{jjk}^i$ (3.27)

That is, the divergences of the electromagnetic field tensor is no more equal to zero, (3.27) becomes the well know Maxwell equation if and only if $\Gamma_{jk}^i = 0$.

Equation (3.26) and (3.27) gives the interaction between the electromagnetic field and the gravitational field.

The other two Maxwell equations follows from (1.20), using (1.19) i.e.:

$$\frac{\partial f_{ij}}{\partial x^k} + \frac{\partial f_{jk}}{\partial x^i} + \frac{\partial f_{ki}}{\partial x^j} = 0 \quad (3.28)$$

Which correspond to (3.17) and (3.18) .

IV. Conclusions

It is very important to recognize the affine connection L^i_{jk} as a real physical field, a unique field; this field contains the gravitational field and the electromagnetic field and seems to fill up all the space.

The equation (3.26) shows that the gradient of the electromagnetic field can be changed if we change the metric of the space. The gradient of the gravitational field can be changed if we change the gradient of the electromagnetic field

Equation (3.27) show that Maxwell's equations, the first two, must be modified in order to take in account the gravitational field, and the interaction between the electromagnetic field and the gravitational field. Equation (1.16) shows the existence of a field of Lie groups.

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