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The Physical Meaning of Einstein Metric and New Derivation of Heisenberg Time and Spatial Evolution

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Abstract: *The unitary operator in the Heisenberg picture and the spatial evolution of the quantum system was found by using simple mathematics and the ordinary laws of differentiation and integration. This expression describes successfully the spatial evolution of the quantum operators. The metrics in the curved space is found to be related to the Lorentz transformation coefficient.*

Key words: *momentum operator, spatial evolution, unitary operator, Einstein metric, Lorentz transformation*

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I. Introduction

Quantum systems are described by operators and wave functions. In the Schrodinger representation the evolution of the quantum system is described by the wave function. In the Heisenberg picture the evolution is described by operators [1, 2]. The change from the Schrodinger to the Heisenberg picture time evolution is done by using mathematical transformation. Schrodinger picture is needed for the probability distribution, while the Heisenberg picture time evolution is needed for the quantum average of the Physical quantity [3, 4]. This transformation is different from the Lorentz transformation which aims to find the effect of motion and fields on the Physical quantities [5, 6]. The quantum system is described by these transformation services successfully. However the spatial evolution of the quantum system is not fully supported. Some attempts have been made to derive Heisenberg spatial evolution of the quantum system [7, 8, and 9] but it needs to be done by another simple approach.

New Derivation of Heisenberg Quantum Equation

In the Heisenberg picture time evolution of the system is described by operators instead of wave function. Thus one needs changing Schrodinger equation

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = \hat{H}|\psi\rangle \qquad (1)$$

Where

$$\hat{H} = \hat{H}_0 + \hat{H}_i \qquad (2)$$

To shift the time dependence from $\psi(t)$ to the operator $\hat{O}$.

This can be achieved by defining

$$|\psi(t)\rangle_s = |\psi(t)\rangle = U(t)|\psi(0)\rangle = U(t)|\psi\rangle_0 = U|\psi\rangle_H \qquad (3)$$

Where the subscript s stand for Schrodinger picture, while H is stands for Heisenberg picture. Thus the wave function in Schrodinger and Heisenberg are related according to the above equation (3). It is clear that $\psi_H$is time dependent while ψ_s is time dependent. To find Heisenberg equation of motion one uses the fact that the expectation values in all representations are equal. This means that they are equal in both Heisenberg and Schrodinger picture, i.e.

$$\langle\psi|_H \hat{O}_H |\psi\rangle_H = \langle\psi|_s \hat{O}_s |\psi\rangle_s \qquad (4)$$

Where $\hat{O}_H$and $\hat{O}_s$ stands for the operators in Heisenberg and Schrodinger picture. Using relation (4) in (3) yields

$$\langle\psi|_H \hat{O}_H |\psi\rangle_H = \langle\psi|_H Ц^{-1} O_s\, Ц\, |\psi\rangle_H \qquad (5)$$

Thus

$$\hat{O}_H = Ц^{-1} O_s\, Ц \qquad (6)$$

Equation (1) can be solved to get

$$\int \frac{d|\psi\rangle}{d|\psi\rangle} = \int \frac{\hat{H}}{i\hbar} dt$$

$$Lnd|\psi\rangle = -\frac{i}{\hbar}\int \hat{H} dt + c_0$$

$$|\psi\rangle = c_1 e^{-\frac{i}{\hbar}\int \hat{H} dt} = e^{-\frac{i}{\hbar}\int \hat{H} dt} c_1$$

But at $t = 0$

$$|\psi(t)\rangle = |\psi(0)\rangle = |\psi\rangle_0 = c_1$$

Thus

$$|\psi\rangle = e^{-\frac{i}{\hbar}\int \hat{H} dt} |\psi\rangle_0 \qquad (7)$$

Comparing this equation (7) with equation (3), yields

$$Ц = e^{-\frac{i}{\hbar}\int \hat{H} dt} = e^{-\frac{i}{\hbar} g}$$

$$Ц^{-1} = e^{-\frac{i}{\hbar} g} \qquad (8)$$

Where

$$g = \int \frac{dg}{dt} dt = \int H dt$$

$$\frac{dg}{dt} = H \qquad (9)$$

This equation (9) together with equation (6) can be used to obtain the time evolution equation of the Heisenberg operator.

Therefore, ones get:

$$\frac{do_H}{dt} = \frac{d}{dt}\left[Ц^{-1} O_s\, Ц\right] = \frac{d}{dt}\left[e^{\frac{i}{\hbar}g} O e^{-\frac{i}{\hbar}g}\right]$$

$$= \frac{i}{\hbar}\frac{dg}{dt} e^{\frac{ig}{\hbar}} O e^{-\frac{i}{\hbar}g} + e^{\frac{i}{\hbar}g}\frac{dO}{dt} e^{-\frac{i}{\hbar}g} - \frac{i}{\hbar} e^{\frac{ig}{\hbar}} O e^{-\frac{i}{\hbar}g}\frac{dg}{dt}$$

$$\frac{i}{\hbar}\left[HO_H - O_H H\right] + \left(\frac{dO_s}{dt}\right)_H$$

$$\frac{dO_H}{dt} = \frac{i}{\hbar}\left[H, O_H\right] + \left(\frac{dO}{dt}\right)_H \qquad (10)$$

On other

$$\left(\int \hat{H}\, dt\right)|\psi_0\rangle = \int i\hbar \frac{d}{dt} dt |\psi\rangle_0$$

$$= i\hbar \int d|\psi\rangle_0 = i\hbar |\psi\rangle_0 \qquad (11)$$

But the same result (11) can be found if one proposes that

$$\hat{H} = i\hbar \frac{d}{dt}$$

To be out of the integration sign to get

$$\int \hat{H} dt\, |\psi\rangle_0 = \hat{H}\int dt\, |\psi\rangle_0 = i\hbar \frac{d}{dt}\int dt |\psi_0\rangle$$

$$=i\hbar\frac{d}{dt}t|\psi\rangle_0 = i\hbar|\psi\rangle_0 \qquad (12)$$

Thus comparing (11) and (12) yields

$$\int \hat{H}\,dt = \hat{H}\int dt = i\hbar$$

$$\hat{H}t = i\hbar \qquad (13)$$

Thus from (8), (7) and (13) at $|\psi\rangle = |\psi\rangle_0 t = 0$

$$Ц_0 = Ц(t=0) = e^{-\frac{i}{\hbar}\int Hdt}e^0 = I \qquad (14)$$

New Derivation of Heisenberg Special Evolution

In this work a new trend based on the unitary operator which is found using some simple mathematical techniques based on the ordinary differentiation and integration is used. The quantum average of the momentum operator in the Schrodinger and Heisenberg picture is also used. Acting on the wave function by the momentum operator

$$\frac{\hbar}{i}\frac{d}{dx}|\psi(x)\rangle = \hat{P}|\psi(x)\rangle \qquad (15)$$

The unitary operator is defined by

$$|\psi(x)\rangle = |\psi(x)\rangle_s = Ц(x)|\psi(0)\rangle = Ц(x)|\psi\rangle_0 = Ц|\psi\rangle_0 = Ц|\psi\rangle_H \qquad (16)$$

The quantum average of the operator O is equal in the Schrodinger and Heisenberg picture. Therefore

$$\langle\psi|_H O_H|\psi\rangle_H = \langle\psi|_s O_s|\psi\rangle_s$$

$$\langle\psi|_H O_H|\psi\rangle_H = \langle\psi|_H Ц^{-1}O_s Ц|\psi\rangle_H \qquad (17)$$

Hence the operator in the Heisenberg picture is given by

$$O_H = Ц^{-1}O_s Ц \qquad (18)$$

$$\int\frac{d|\psi\rangle}{|\psi\rangle} = \frac{i}{\hbar}\int\frac{iP}{\hbar}dx$$

$$Ln|\psi\rangle = \frac{i}{\hbar}\int Pdx + C_2$$

$$|\psi\rangle = e^{\frac{i}{\hbar}\int Pdx}C_2 \qquad (19)$$

But at

$x = 0 \qquad |\psi(x)\rangle = |\psi(0)\rangle = C_2 \quad (20)$

Thus equations (19) and (16) give

$$|\psi\rangle = e^{\frac{i}{\hbar}\int P(x)dx}|\psi\rangle_0 = e^{\frac{i}{\hbar}f(x)}|\psi\rangle_0 = e^{\frac{i}{\hbar}f(x)}|\psi\rangle_H \qquad (21)$$

Where

$$f(x) = \int\frac{df}{dx}dx = \int P(x)dx$$

Thus

$$P = \frac{df}{dx} \qquad (22)$$

Hence equations (2) and (7) can be compared to get

$Ц = e^{\frac{i}{\hbar}\int Pdx} = e^{\frac{i}{\hbar}f(x)} = e^{\frac{i}{\hbar}f}$ (23)

Thus equation (4) is given by

$$O_H = e^{-\frac{i}{\hbar}f(x)} O_s e^{\frac{i}{\hbar}f}$$

$= e^{-\frac{i}{\hbar}f} O e^{\frac{i}{\hbar}f}$ (24)

The spatial evolution of the operator is therefore given by:

$$\frac{dO_H}{dx} = -\frac{i}{\hbar}\frac{df}{dx}e^{-\frac{i}{\hbar}f}Oe^{\frac{i}{\hbar}f} + e^{-\frac{i}{\hbar}f}\frac{dO}{dx}e^{\frac{i}{\hbar}f} + \frac{i}{\hbar}e^{-\frac{i}{\hbar}f}Oe^{\frac{i}{\hbar}f}\frac{df}{dx}$$

$= -\frac{i}{\hbar}[PO_H - O_H P] + \left(\frac{dO}{dx}\right)_H$

$\frac{\hbar}{i}\frac{dO_H}{dx} = [O_H, P] + \left(\frac{dO}{dx}\right)_H$ (25)

On the other hand

$$\int Pdx\,|\psi\rangle_0 = \int \frac{\hbar}{i}\frac{d}{dx}dx|\psi\rangle_0$$

$\int Pdx\,|\psi\rangle_0 = \int d\,|\psi\rangle_0 = \frac{\hbar}{i}|\psi\rangle_0$ (26)

The same result can be obtained by suggesting that

$$P = \frac{\hbar}{i}\frac{d}{dx}$$

To be out of integration to get

$$\int Pdx\,|\psi\rangle_0 = P\int dx\,|\psi\rangle_0 = \frac{\hbar}{i}\frac{d}{dx}(x)|\psi\rangle_0 = \frac{\hbar}{i}|\psi\rangle_0$$

Thus

$\int Pdx = P\int dx = \frac{\hbar}{i}$ (27)

From (9), when

$$|\psi\rangle = |\psi\rangle_0 x = 0$$

$Ц\;(x=0) = Ц_0 = e^{\frac{i}{\hbar}\int Pdx} = e^0 = I$ (28)

Special and General Relativistic Meaning of the Matrix

The Proper length or proper time I a relativistic space-time takes the general form

$c^2 d\tau^2 = -g_{\mu v}dx^{\mu}dx^{v} = -g'_{\mu v}dx^{\mu'}dx^{v'}$ (29)

Consider two space points in the frame s and s' that measured simultaneously. In this case

$$dx^0 = 0 \qquad dx^{0'} = 0$$

Therefore

$g_{xx}dx^2 = g'_{xx}dx'^2$ (30)

An observer at s observe a rod which is at rest in s' moving with constant speed υ. So:

$g_{xx} = 1 \qquad g'_{xx} = 1 - v^2/c^2$(31)

To get the ordinary length contraction relation

$$dx = \sqrt{g'_{xx}}dx'$$

$dx = \sqrt{1 - v^2/c^2}\,dx'$ (32)

When s' moves with speed of light

$dt' = 0 \qquad \dot{x}' = c^2$ (33)

Thus from (29)

$$-c^2dt^2 - g_{xx}dx^2 = -g_{xx}^{'}dx^{'2} \quad (34)$$

$$-c^2 - v^2 g_{xx} = -c^2 g_{xx}^{'}$$

Thus since

$$g_{xx} = -1$$

$$g_{xx}^{'} = (1 - v^2/c^2) \quad (35)$$

Consider also a clock at rest at a certain point in$s^{'}$. In this case one must use the time metric relation to get

$$dx^0 = \sqrt{g_{00}^{'}}dx^{0'}$$

$$dt = (1 - v^2/c^2)^{-\frac{1}{2}}dt^{'} \; (36)$$

$$=(1 - v^2/c^2)^{-\frac{1}{2}}dt_0$$

It is very interesting to note that

$$g_{xx} = g_{00}^{-1} \quad (37)$$

This conforms to Schwarzschild solution. Thus in a curved space time
(Curved ≡C) which is equivalent to an accelerated frame $s^{'}$ with respect to an observer which is at rest in s

$$dx_c = \sqrt{g_{xx}^{'}}dx^{'}(38)$$

II. Discussion

Using the quantum average of the Hamiltonian of the quantum system in equation (4) and the definition of the unitary operator in (6) one finds the expression of the unitary operator in the Heisenberg time evolution equation in (10). Using the quantum average of the momentum operator of the quantum system in the Schrodinger and Heisenberg picture in equation (17) beside integration and differentiation technique, one finds the functional form of the unitary operator defined in equation (16).. Then the Heisenberg spatial evolution is found in equation (25).The proper length in a curved space is written in equation (30). Comparing this expression with the corresponding Special relativity expressions, one finds that the spatial and time metric is related to the Lorentz transformation factor as shown in equations (31) & (36).They satisfy Schwarzschild relation.

III. Conclusion

The spatial evolution of the quantum system is found using unitary operator and simple mathematics based on the ordinary differentiation and integration. The metrics in the curved space time is found to be related to the Lorentz transformation factor.

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