IOSR Journal of Applied Physics (IOSR-JAP)
e-ISSN: 2278-4861.Volume 11, Issue 5 Ser. I (Sep. –Oct. 2019), PP 31-38
www.iosrjournals.org

Space-Time Evolution ofone-Dimensional Dirac Wave Packet

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Abstract: *We present a brief description of a stationary localized state of arelativistic free particle derived from a Dirac equation with an interaction in one space dimension. This localized state evolves according to the original Dirac equation when the interaction is switched off. The time evolution of the wave packet is found analytically which shows clear asymmetry between the two components that move opposite to each other. There is also an interference part similar to zitterbewegung.*

Date of Submission: 03-09-2019 Date of Acceptance: 18-09-2019

I. Introduction

A wave packet can be a stationary state as shown in Refs. [1-2]. The referred findings are in the domainof nonrelativistic quantum mechanics. Recently, we introduced an interaction the coupling of which with the free Dirac Hamiltonian results in stationary localized wave packet states of a relativistic particle [3]. Moreover, study of relativistic wave packet is by itself important and as such many authors devoted time to its study (see, for example, Ref. [4] and the references therein). In this brief report, we begin with the creation of a localized wave packet which is a stationary state of the Dirac equation

$$[c\vec{\alpha}.(\vec{p}-iq\vec{r})+\beta mc^2]\psi = E\psi\ , \tag{1}$$

where, the interaction term $(-iq\vec{r})$ is added to produce states in one dimension as of the form

$$\psi_{up}(z) = N\begin{pmatrix}1\\0\\1\\0\end{pmatrix}\exp\left(-\frac{qz^2}{2\hbar}\right)\exp(ik_0 z) \tag{2}$$

Here, $k_0 = \frac{1}{\hbar}\sqrt{\frac{E^2}{c^2}-m^2c^2}$ is the wave number, m is the mass of the Dirac particle, say, electron, q has the dimension of Kgs^{-1} and will be called the envelope parameter, and $N=\sqrt{\frac{1}{2}\sqrt{\frac{q}{4\pi\hbar}}}$ is the normalization constant, and all other symbols carry usual meanings. Equation (2) represents spin $\frac{1}{2}$ particle with spin aligned in the $(+z)$ direction. Equation (2) is plotted in Figure (1) for only the top component. Clearly it is a localized wave packet.

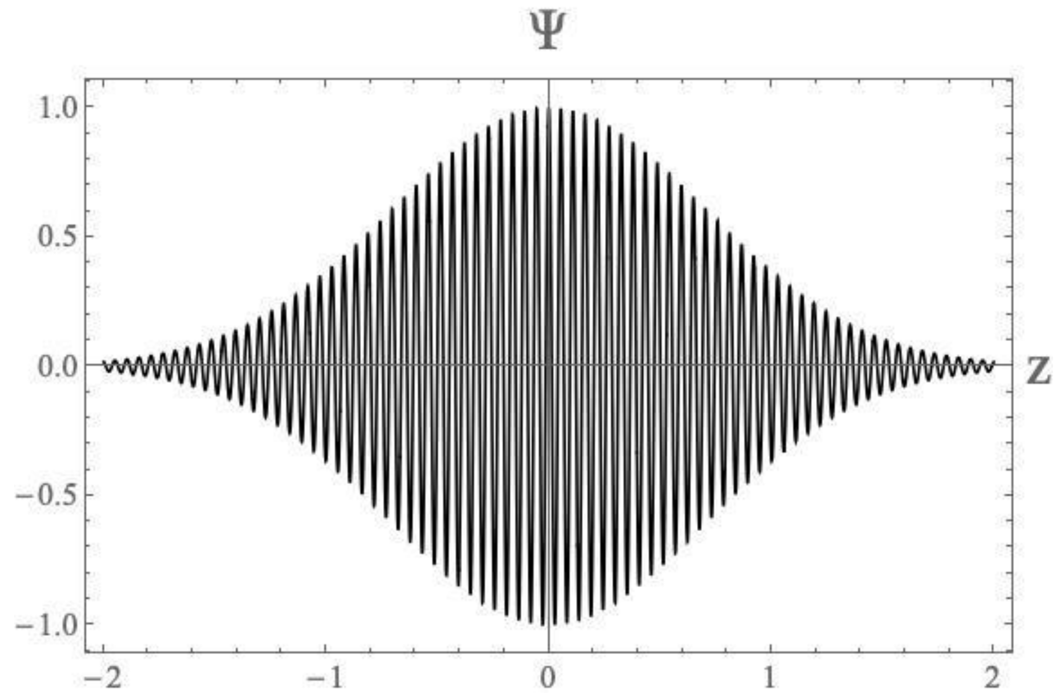

Figure 1: The un-normalized rendering of the top component of Eq. (2). Here $k_0 = 116\pi$ and $q/_{2\hbar} = 1$.Clearly, this is a localized wave packet.

For spin in the $(-z)$ direction, the wave packet is

$$\psi_{down}(z) = N\begin{pmatrix}0\\1\\0\\1\end{pmatrix}\exp\left(-\frac{qz^2}{2\hbar}\right)\exp(ik_0 z) \quad (3)$$

Now, when the interaction $-(iq\vec{r})$ is switched off, the state (2) and (3) will evolve according to the general theory of time evolution.

In this paper, we shall work out this time evolution and examine the probability density in space and time and for that matter, we shall use the standard procedure outlined in [5-7]. In the next section, we present our calculation and in the final section, we shall summarize our result and discuss them for further use.

II. Time evolution of wave packet in one space dimension

Using the standard procedure from Ref. [5], we can write the general time-independent solution of the free Dirac equation as

$$\psi(z) = \int_{-\infty}^{\infty}\sum_r C_r(p)\psi_r(z)dp \quad (4)$$

where, $\psi_r(z) = \varphi_p(z)U_r(p)$, $r = 1, 2, 3, 4,$

and

$$\varphi_p(z) = \frac{1}{\sqrt{2\pi\hbar}}e^{(ipz/\hbar)}. \quad (5)$$

Now,

$$U_1(p) = N\begin{pmatrix} 1 \\ 0 \\ p\gamma \\ 0 \end{pmatrix} \tag{6}$$

$$U_2(p) = N\begin{pmatrix} 0 \\ 1 \\ 0 \\ -p\gamma \end{pmatrix} \tag{7}$$

$$U_3(p) = N\begin{pmatrix} -p\gamma \\ 0 \\ 1 \\ 0 \end{pmatrix} \tag{8}$$

$$U_4(p) = N\begin{pmatrix} 0 \\ p\gamma \\ 0 \\ 1 \end{pmatrix} \tag{9}$$

and

$$\gamma = \frac{c}{E_p + mc^2}\,. \tag{10}$$

Here U_1 and U_2 correspond to energy $E = E_p$ and U_3 and U_4 correspond to the energy $E = -E_p$,

$N = \sqrt{\frac{E_p + mc^2}{2E_p}}$ and $E_p = \sqrt{p^2c^2 + m_0^2c^4}$.

Now, $C_r(p)$ are given by the Fourier transformation [5] as

$$C_r\left(p\right) = \int_{-\infty}^{\infty} \psi_r^{\dagger}\left(z\right)\psi\left(z\right)dz \tag{11}$$

The time evolution of $\Psi(z)$ is given by [5] $\Psi(z,t)$ as follows:

$$\psi(z,t) = \int_{-\infty}^{\infty} \sum_r C_r(p)\psi_r(z)e^{-iE_r t/\hbar} \tag{12}$$

where, $E_r = +E_p$ for $r = 1, 2$ and $E_r = -E_p$ for $r = 3, 4$.Evaluating $C_r(p)$ by Eq. (11) using $\Psi(z)$ given by Eq. (2) and writing $\psi(z,t) = \begin{pmatrix}\psi_1 & \psi_2 & \psi_3 & \psi_4\end{pmatrix}^T$, we obtain $\psi_2 = \psi_4 = 0$, by virtue of the free Dirac solutions for constant unidirectional velocity of a fermion, which are

$$\psi_1(z) = \frac{1}{\sqrt{2\pi\hbar}} N e^{\frac{ipz}{\hbar}} \begin{pmatrix} 1 \\ 0 \\ \frac{p}{E_p + m} \\ 0 \end{pmatrix}, \quad E = E_p \tag{13}$$

$$\psi_2(z) = \frac{1}{\sqrt{2\pi\hbar}} N e^{\frac{ipz}{\hbar}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -\frac{p}{E_p + m} \end{pmatrix}, \quad E = E_p \tag{14}$$

$$\psi_3(z) = \frac{1}{\sqrt{2\pi\hbar}} N e^{\frac{ipz}{\hbar}} \begin{pmatrix} -\frac{p}{E_p + m} \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad E = -E_p \tag{15}$$

$$\psi_4(z) = \frac{1}{\sqrt{2\pi\hbar}} N e^{\frac{ipz}{\hbar}} \begin{pmatrix} 0 \\ \frac{p}{E_p + m} \\ 0 \\ 1 \end{pmatrix}, \quad E = -E_p \tag{16}$$

These states are taken from Ref. [8] which corresponds to free particle moving with fixed momentum$p$.
Then, as we will see C_2 and C_4 both are zero, we obtain

$$\psi(z,t) = \int_{-\infty}^{\infty} C_1 \psi_1(z)\ e^{-iE_1 t/\hbar} dp + \int_{-\infty}^{\infty} C_3 \psi_3(z)\ e^{-iE_3 t/\hbar} dp \tag{17}$$

Now,

$$C_1 = \int_{-\infty}^{\infty} \psi_1^{\dagger}(z) \psi(z) dz \tag{18}$$

$$C_2 = \int_{-\infty}^{\infty} \psi_2^{\dagger}(z) \psi(z) dz \tag{19}$$

$$C_3 = \int_{-\infty}^{\infty} \psi_3^{\dagger}(z) \psi(z) dz \tag{20}$$

$$C_4 = \int_{-\infty}^{\infty} \psi_4^{\dagger}(z) \psi(z) dz \tag{21}$$

For brevity, we will show only the evaluation ofC_1, where

$$C_1(p) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{-ipz}{\hbar}} \begin{pmatrix} 1 & 0 & \frac{p}{E_p + m} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} e^{ik_0 z} e^{-\frac{qz^2}{2\hbar}} \sqrt{\frac{1}{2}\sqrt{\frac{q}{4\pi\hbar}}} \sqrt{\frac{E_p + m}{2E_p}} dz$$

$$= \frac{1}{2}\sqrt{\frac{E_p + m}{2E_p}} \sqrt[4]{\frac{1}{q\pi\hbar}} \left(1 + \frac{p}{E_p + m}\right) \exp\left\{-\frac{\hbar}{2q}\left(k - k_0\right)^2\right\}, \quad p = \hbar k\ . \tag{22}$$

In the above expression, we have set,c, the speed of light in free space, to unity.
Then, we arrive at

$$\psi\left(z,t\right) = \int_{-\infty}^{\infty} C_1 \psi_1\left(z\right) e^{-iE_1 t/\hbar} dp + \int_{-\infty}^{\infty} C_3 \psi_3\left(z\right) e^{-iE_3 t/\hbar} dp \tag{23}$$

Now, we employ the following structure:

$$\psi(z,t)=\begin{pmatrix}\psi_1(z,t)\\ \psi_2(z,t)\\ \psi_3(z,t)\\ \psi_4(z,t)\end{pmatrix} \quad (24)$$

After a lengthy calculation, we arrive at,

$$\psi_1(z,t)=\frac{1}{\sqrt{8\pi\hbar}}\sqrt[4]{\frac{1}{q\pi\hbar}}\int_{-\infty}^{\infty}(\frac{E_p+m}{2E_p})(1+\frac{p}{E_p+m})\exp\{-\frac{\hbar}{2q}(k-k_0)^2\}exp\{i(kz-\frac{E_pt}{\hbar})\}dp$$
$$+\frac{1}{\sqrt{8\pi\hbar}}\sqrt[4]{\frac{1}{q\pi\hbar}}\int_{-\infty}^{\infty}(\frac{E_p+m}{2E_p})(1-\frac{p}{E_p+m})\exp\{-\frac{\hbar}{2q}(k-k_0)^2\}exp\{i(kz+\frac{E_pt}{\hbar})\}(-\frac{p}{E_p+m})dp \quad (25)$$

Because, $C_2=C_4=0,\ \psi_2(z,t)=\psi_4(z,t)=0$.
We obtain for $\Psi_3(z,t)$ as follows:

$$\psi_3(z,t)=\frac{1}{\sqrt{8\pi\hbar}}\sqrt[4]{\frac{1}{q\pi\hbar}}\int_{-\infty}^{\infty}(\frac{E_p+m}{2E_p})(1+\frac{p}{E_p+m})\exp\{-\frac{\hbar}{2q}(k-k_0)^2\}exp\{i(kz-\frac{E_pt}{\hbar})\}(\frac{p}{E_p+m})dp$$
$$+\frac{1}{\sqrt{8\pi\hbar}}\sqrt[4]{\frac{1}{q\pi\hbar}}\int_{-\infty}^{\infty}(\frac{E_p+m}{2E_p})(1-\frac{p}{E_p+m})\exp\{-\frac{\hbar}{2q}(k-k_0)^2\}exp\{i(kz+\frac{E_pt}{\hbar})\}dp \quad (26)$$

$\psi_1(z,t)$ can be simplified, because of the integration over p from $-\infty$ to $+\infty$, to

$$\psi_1(z,t)=\frac{1}{\sqrt{8\pi\hbar}}\sqrt[4]{\frac{1}{q\pi\hbar}}\int_{-\infty}^{\infty}(\frac{E_p+m}{2E_p})\exp\{-\frac{\hbar}{2q}(k-k_0)^2\}exp\{i(kz-\frac{E_pt}{\hbar})\}dp$$
$$+\frac{1}{\sqrt{8\pi\hbar}}\sqrt[4]{\frac{1}{q\pi\hbar}}\int_{-\infty}^{\infty}(\frac{E_p-m}{2E_p})\exp\{-\frac{\hbar}{2q}(k-k_0)^2\}exp\{i(kz+\frac{E_pt}{\hbar})\}dp \quad (27)$$

Similarly, $\psi_3(z,t)$ becomes

$$\psi_3(z,t)=\frac{1}{\sqrt{8\pi\hbar}}\sqrt[4]{\frac{1}{q\pi\hbar}}\int_{-\infty}^{\infty}(\frac{E_p-m}{2E_p})\exp\{-\frac{\hbar}{2q}(k-k_0)^2\}exp\{i(kz-\frac{E_pt}{\hbar})\}dp$$
$$+\frac{1}{\sqrt{8\pi\hbar}}\sqrt[4]{\frac{1}{q\pi\hbar}}\int_{-\infty}^{\infty}(\frac{E_p+m}{2E_p})\exp\{-\frac{\hbar}{2q}(k-k_0)^2\}exp\{i(kz+\frac{E_pt}{\hbar})\}dp \quad (28)$$

In the non-relativistic limit, where $E_p\approx m$,

$$\psi_1(z,t)=\frac{1}{\sqrt{8\pi\hbar}}\sqrt[4]{\frac{1}{q\pi\hbar}}\int_{-\infty}^{\infty}\exp\{-\frac{\hbar}{2q}(k-k_0)^2\}exp\{i(kz-\frac{E_pt}{\hbar})\}dp \quad (29)$$

And

$$\psi_3(z,t)=\frac{1}{\sqrt{8\pi\hbar}}\sqrt[4]{\frac{1}{q\pi\hbar}}\int_{-\infty}^{\infty}\exp\{-\frac{\hbar}{2q}(k-k_0)^2\}exp\{i(kz+\frac{E_pt}{\hbar})\}dp \quad (30)$$

In the ultra-relativistic limit, $E_p\gg m$,

$$\psi_1(z,t) = \frac{1}{\sqrt{8\pi\hbar}}\sqrt[4]{\frac{1}{q\pi\hbar}}\int_{-\infty}^{\infty}\frac{1}{2}\exp\{-\frac{\hbar}{2q}(k-k_0)^2\}exp\{i(kz-\frac{E_p t}{\hbar})\}dp + \frac{1}{\sqrt{8\pi\hbar}}\sqrt[4]{\frac{1}{q\pi\hbar}}\int_{-\infty}^{\infty}\frac{1}{2}\exp\{-\frac{\hbar}{2q}(k-k_0)^2\}exp\{i(kz+\frac{E_p t}{\hbar})\}dp \quad (31)$$

And in this limit,

$$\psi_3(z,t) = \frac{1}{\sqrt{8\pi\hbar}}\sqrt[4]{\frac{1}{q\pi\hbar}}\int_{-\infty}^{\infty}\frac{1}{2}\exp\{-\frac{\hbar}{2q}(k-k_0)^2\}exp\{i(kz-\frac{E_p t}{\hbar})\}dp + \frac{1}{\sqrt{8\pi\hbar}}\sqrt[4]{\frac{1}{q\pi\hbar}}\int_{-\infty}^{\infty}\frac{1}{2}\exp\{-\frac{\hbar}{2q}(k-k_0)^2\}exp\{i(kz+\frac{E_p t}{\hbar})\}dp \quad (32)$$

For the general case, we compute the integrals involved using $E_p = \hbar\omega$ and $\omega(k) = \omega_0 + (k-k_0)v_g$.
To the first approximation,

$$\int_{-\infty}^{\infty}\exp\{-\frac{\hbar}{2q}(k-k_0)^2\}exp\{i(kz-\omega t)\}dk = \sqrt{\frac{2q\pi}{\hbar}}\exp[-\frac{q}{2\hbar}(z-v_g t)^2]\exp\{i(k_0 z-\omega_0 t)\}$$

where, we have used, $dp = \hbar dk$. (33)
From (29), we obtain,

$$\psi_1(z,t) = \frac{\hbar}{\sqrt{8\pi\hbar}}\sqrt[4]{\frac{1}{q\pi\hbar}}\sqrt{\frac{2q\pi}{\hbar}}\exp[-\frac{q}{2\hbar}(z-v_g t)^2]exp\{i(k_0 z-\omega_0 t)\}$$

$$= \sqrt[4]{\frac{q}{16\pi\hbar}}\exp[-\frac{q}{2\hbar}(z-v_g t)^2]exp\{i(k_0 z-\omega_0 t)\} \quad (34)$$

in the non-relativistic limit. And in the same limit,

$$\psi_3(z,t) = \sqrt[4]{\frac{q}{16\pi\hbar}}\exp[-\frac{q}{2\hbar}(z+v_g t)^2]exp\{i(k_0 z+\omega_0 t)\} \quad (35)$$

In between the ultra-relativistic limit and the non-relativistic regime, i.e. in generally relativistic case, we have to evaluate Eq. (27) and (28). So, if we now do the integrations assuming for $\frac{E_p + mc^2}{2E_p} \approx \frac{\gamma_0 + 1}{2\gamma_0}$, where γ_0 is the relativistic γ-factor evaluated at k_0,
and $E_p = \hbar\omega \approx \hbar[\omega_0 + (k-k_0)\frac{d\omega}{dk} + ...] = \hbar[\omega_0 + (k-k_0)v_g]$, to the first approximation,
then, we obtain for relativistic case in general

$$\psi_1(z,t) = \sqrt[4]{\frac{q}{16\pi\hbar}}[\left(\frac{\gamma_0+1}{2\gamma_0}\right)\exp[-\frac{q}{2\hbar}(z-v_g t)^2]exp\{i(k_0 z-\omega_0 t)\} + \left(\frac{\gamma_0-1}{2\gamma_0}\right)\exp[-\frac{q}{2\hbar}(z+v_g t)^2]exp\{i(k_0 z+\omega_0 t)\}] \quad (36)$$

and

$$\psi_3(z,t) = \sqrt[4]{\frac{q}{16\pi\hbar}}[\left(\frac{\gamma_0-1}{2\gamma_0}\right)\exp[-\frac{q}{2\hbar}(z-v_g t)^2]exp\{i(k_0 z-\omega_0 t)\} + \left(\frac{\gamma_0+1}{2\gamma_0}\right)\exp[-\frac{q}{2\hbar}(z+v_g t)^2]exp\{i(k_0 z+\omega_0 t)\}] \quad (37)$$

The probability density $P(z,t) = \Psi^{\dagger}\Psi$ is given by

$$P(z,t)=\sqrt{\frac{q}{16\pi\hbar}}[\left(\frac{\gamma_0+1}{2\gamma_0}\right)^2\exp\{-\frac{q}{\hbar}(z-v_g t)^2\}+\left(\frac{\gamma_0-1}{2\gamma_0}\right)^2\exp\{-\frac{q}{\hbar}(z+v_g t)^2\}$$
$$+\left(\frac{\gamma_0{}^2-1}{4\gamma_0{}^2}\right)\exp\{-\frac{q}{\hbar}(z-v_g t)^2-\frac{q}{\hbar}(z+v_g t)^2\}\cos(2\omega_0 t)] \tag{38}$$

Hence, we see two Gaussian profiles given by the first two terms of Eq. (38) to move along $(+z)$and $(-z)$ direction with unequal amplitude and an interference pattern given by the 3rd term filling the whole space, but this diminishes as time goes on. This interference term, we suppose, is the zitterbewegung discovered long ago [5]. Hence, we have obtained a complete account of evolution in time of a Dirac wave packet, although the Dirac packet here is not the usual one. We found the original $\Psi(z)$ from the modified Eq. (1). However, the $\Psi(z)$ in our Eq. (2) is very much similar to the one given in Eq. (24) of Ref. [5]. Hence, the beginning of our calculation is in accord with what we find in literature and our end result is also in accord with what we see in literature (see for example, Ref. [5]). As far as our knowledge goes, the treatment presented here has not been reported elsewhere for one-dimensional wave packet of any Dirac equation.

To visualize what we have found in Eq. (38), we plot the first two terms at particular values of$z,\ t, q$ in Fig. (2). Clearly, one Gaussian moves toward left with smaller amplitude than the Gaussian that moves along right. Several plots are given to internalize the meaning of Eq. (38). The interference term is also plotted in Fig. (2).

Our work in this paper is, in our view, an important contribution to relativistic quantum mechanics. This will generate many more insights about wave packet. Certainly, the wave packet we have found for relativistic, non-relativistic and ultra-relativistic cases, they all suffer dispersion. But looking at the center of the wave packet, which we have done, gives also many insights. If the energy-momentum relation is for that of mass-less particle, as electrons in graphene, the wave packets will not suffer dispersion as is easily evident from our derivation. Thus, we have presented a result pertaining to relativistic quantum mechanics. This will be of help in quantum field theory, too and will open up a door to enter a vast area of physics.

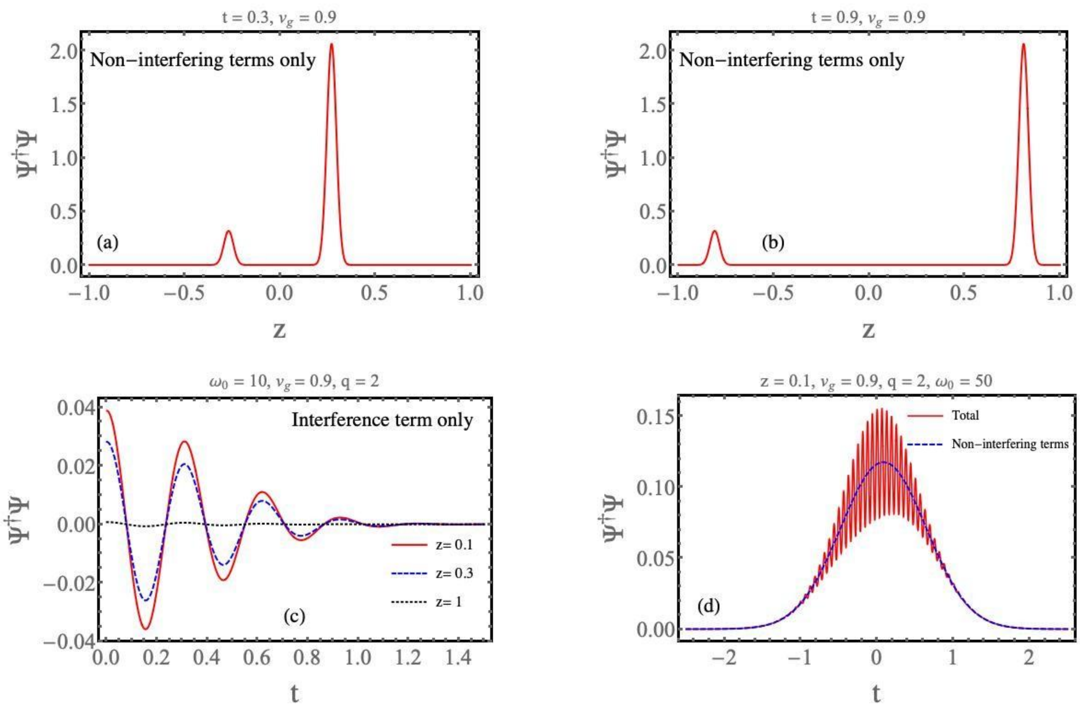

Figure 2: Plot of un-normalized probability density with $v_g = 0.9$ and, $c = \hbar = 1$. (a) The two non-interfering terms with $t = 0.3$and $q = 256\pi$. (b) The two non-interfering terms with $t = 0.9$and $q = 256\pi$. (c) The interference term alone, with $\omega_0 = 10, q = 2$. (d) Effect of the interference term in the total probability density.

III. Discussion and Conclusion:

In this paper, we have utilized the equation we have published in Ref. [3].

The solution of that equation is very interesting, namely, stationary and localized wave packet. These wave packets have long been used in literature, but from where they appear is not clear. What is known by all is that superposition of many free solutions yield a wave packet. We have firstly shown the origin of localized wave packets from a modified Dirac equation. This Dirac equation is P-T symmetric.Without changing the energy spectrum, it gives states which are confined and oscillating rapidly. However, we did not stop there. We have found out the time evolution of the wave packets and found excellent agreement with results in literature for other type of initial localized wave packet [5]. A complete treatment for the case of relativistic one-dimensional wave packet-time-evolution was somewhat lacking in literature, but our results will cover up that empty space.

In short, the one dimensional wave packet has two prongs, one with larger amplitude and moving to the $(+z)$ direction and the other is of slightly smaller amplitude and moving to the $(-z)$ direction. Although, we have neglected dispersion by assuming for $\omega(k)$ a first order Taylor series given by $\omega(k) = \omega_0 + (k - k_0)v_g$, our result gives a fairly accurate picture of time evolution of one-dimensional wave packet. One may object that the initial wave packet is not an ordinary one that stems from free Dirac equation, but that objection is not harmful, since in literature almost exactly the same initial wave packet has been considered where our $z-$dependent terms are replaced by $r-$dependent terms (see Eq. (24) of Ref. [5]), but with the same spinor structure. One of the striking features of Dirac wave packets is the trembling motion, called zitterbewegung, which so far has been seen unavoidable. We too could not avoid that. The third term in Eq. (38) of probability density, which we have called interference, is the zitterbewegung. However, although zitterbewegung spans all space in one dimension, it dies out as time goes on. We have visualized our solution in Figure (2) and these are very similar to the 3D case considered in Ref. [5]. Thus, we have reviewed and found some new insights from this work.

Acknowledgement

S.B. Faruque acknowledges the grant PS/2018/2/22 from Research Center of Shahjalal University of Science and Technology, Sylhet, Bangladesh, which supported his work. Promod C. Baidyahappily thanks the Information and CommunicationsTechnology Division under the Ministry of Posts, Telecommunications and Information Technology of the Government of Bangladesh for supporting his work.

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IOSR Journal of Applied Physics (IOSR-JAP) is UGC approved Journal with Sl. No. 5010, Journal no. 49054.

S. B. Faruqu"Space-Time Evolution of one-Dimensional Dirac Wave Packet." IOSR Journal of Applied Physics (IOSR-JAP) , vol. 11, no. 5, 2019, pp. 31-38.