

## **An Extensible Model of the Electron: An Addendum to an Article published by P.A.M. Dirac in 1962**

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**Abstract:** *The primary purpose of this paper is to provide the general solution to the equation describing radial motion for a classical extensible model of an electron as formulated by Dirac (1962). The general solution can be used in a Bohr-Sommerfeld quantization scheme for both angular momentum and vibrations. It is shown that one cannot satisfy both quantum requirements simultaneously but one or the other can always be satisfied exactly. Dirac considered only the vibration quantization (leading to a definition of the muon) from the point of view of small amplitude oscillations around an equilibrium state and so did not obtain an acceptable mass for the muon and he was silent about the angular momentum requirement.*

**Key words:** *electron, extensible model, Dirac*

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### **I. Introduction**

The dynamics of the electron has been the center of many attempts to understand this enigmatic particle. A very detailed review of the history of such attempts is presented by Rohrlich (1997) in which the successes and failures of the historical struggles are clearly and cleanly listed. While there is no winning prescription as yet to providing an understanding of the electron there has been significant progress starting with, perhaps, Poincaré (1906). The present article is not intended to do justice to the plethora of papers concerning electron dynamics but rather is concentrated on one article by Dirac (1962) in order to show that general solutions to equations modeling electron dynamics yield extremely different behaviors than does the approximation of small amplitude oscillations around a stationary equilibrium.

It is now nearly 60 years since Dirac published an article to describe the electron as an oscillating body with the oscillations considered as yielding a muon described through the first excited state of the electron when quantized according to the Bohr-Sommerfeld paradigm. The basic idea was to have an equivalent surface tension to stop the electron from disintegrating due to the repulsive charge on the body. The equation of purely radial motion of the electron radius, including effects due to curvilinear coordinates, was derived by Dirac (1962) (q.v) and included relativistic effects. The exception, as clearly stated by Dirac, was the exclusion of electron spin so that one was treating with a classical electron.

Small amplitude oscillations around an equilibrium radius were considered. The frequency,  $\nu$ , of the oscillations was then taken to correspond to an energy  $h\nu$  and so an equivalent mass;  $M = h\nu/c^2$ , regrettably the equivalent mass turned out to be 448 times the electron mass and so was not acceptable as a representation of a muon. Indeed, Dirac (1962) noted this fact himself and remarked “the one-quantum oscillation is by no means a small one and the approximations used (...) are not valid for it” There was also the problem that for the “small amplitude” oscillations no determination of the amplitude was made- perhaps not a fatal flaw but one was left wondering nevertheless.

Indeed, while the equation of radial motion of the electron radius included relativistic effects, small oscillations around the equilibrium radius did not include such effects so one was again left wondering what was then the point of deriving an equation for motion of the electron radius and then not using its behavior?

A thorough search of Dirac’s (1962) article allows one to note that the fundamental equation of radial motion was *not* solved at all. Speculations as to the reasons for this neglect are legion: such as the equation is nonlinear and so not easy to solve; one had no further interest in the equation of motion when small amplitude oscillations around the equilibrium radius failed to obtain the correct muon mass; etc.

However, as noted by Whitham (1974) (when dealing with construction of exact solutions to nonlinear equations rather than expansions to some small order  $\epsilon$ ) “Not least is the lesson that exact solutions are still around and one should not always turn too quickly to a search for the  $\epsilon$ .”

This article is concerned with determining precisely the exact solutions to the equation of radial motion as given by Dirac (1962). Technical details are presented in the next section of the paper and the exact solutions obtained. The solutions differ markedly from the small oscillation behavior given by Dirac (1962) as was to be

expected after Whitham's (1974) remark. A numerical section then follows illustrating the behavior of the exact solutions for specific parameter values. Based on the exact solutions a short section is then devoted to account for a quantum representation based on the Bohr-Sommerfeld quantization prescription.

### Exact solutions to the radial motion equation

Dirac worked the problem through to the equation of radial motion using a representation where the velocity of light,  $c$ , is identically unity:  $c=1$ . The time varying radius of the electron  $\rho$  then satisfies the equation (Dirac, equation 14)

$$d/dT(w(1-w^2)^{-1/2}) = 2a^3/\rho^4 - 2\rho^{-1}(1-w^2)^{-1/2} \quad (1)$$

where  $w = d\rho/dT$  and the equilibrium radius  $a$  is given through  $a^3 = e^2/4\omega$  and so  $\omega$  measures the fundamental strength of the surface tension just as  $e$  represents the fundamental charge on the electron. In physical units (where  $c \neq 1$ ) and with the representation  $\rho = a\zeta$ , one has the equation in the dimensionless form

$$d/dt(u(1-u^2)^{-1/2}) = 2/\zeta^4 - 2\zeta^{-1}(1-u^2)^{-1/2} \quad (2a)$$

where time  $t$  is measured in units of  $a/c$  so that one has  $u = d\zeta/dt$ . Note that should one wish to have the equilibrium radius to be precisely the classical electron radius  $r_* = e^2/(mc^2)$  then one arranges the surface tension so that  $\omega = (mc^2)^3/(4e^4)$  with  $m$  the electron mass.

Equation (2a) can be solved in full generality as follows. Set  $(1-u^2)^{-1} = A^2$  and then shift to  $\zeta$  instead of  $t$  as a basic variable yielding equation (2a) in the form

$$dA/d\zeta + 2A/\zeta = 2/\zeta^4 \quad (2b)$$

The result is a linear equation in  $A$  with general solution

$$A^2 = \zeta^{-6}(\zeta\Lambda - 2)^2 \quad (3)$$

where  $\Lambda$  is an arbitrary but positive constant of integration. One has  $A = \Lambda\zeta^{-2} - 2\zeta^{-3}$  and the demand  $A > 0$  because  $A = (1-u^2)^{-1/2}$  which must be positive. Thus  $\Lambda > 2\zeta^{-1}$  or  $\zeta > 2\Lambda^{-1}$ . Use the solution (3) and the definition  $(1-u^2)^{-1} = A^2$  to obtain

$$(d\zeta/dt)^2 = 1 - \zeta^6(\zeta\Lambda - 2)^{-2} \quad (4) \text{ which can also be}$$

written in the form

$$(d\zeta/dt)^2 = (\zeta\Lambda - 2 - \zeta^3)(\zeta\Lambda - 2 + \zeta^3)(\zeta\Lambda - 2)^{-2} \quad (5)$$

Because the left hand side of equation (5) is positive then so too must the right hand side be. Thus there is the requirement

$$\zeta\Lambda - 2 - \zeta^3 > 0 \quad (6)$$

which demands  $\Lambda > 3$  with equality ( $\Lambda=3$ ) only on  $\zeta = 1$  corresponding to a fixed radius for the electron at the equilibrium value of  $\rho = a$ . A sketch of the critical curve for  $\Lambda$  versus  $\zeta$  is given in figure 1 showing the domain where physical solutions are possible.

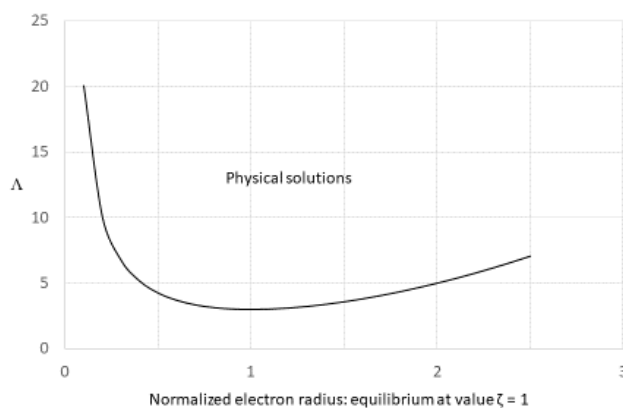


Figure 1

Note for later use that when  $\zeta \ll 1$  an approximate solution of equation (6) is  $\zeta = 2/\Lambda$  for large values of  $\Lambda \gg 1$  while the other approximate solution for  $\Lambda \gg 1$  is  $\zeta = \Lambda^{1/2}$ .

For any value of  $\Lambda > 3$  one has minimum and maximum values of  $\zeta$  ( $\zeta_{\min}$  and  $\zeta_{\max}$  respectively) within which a solution oscillates with time. Figure 2 shows a sketch of such a situation. Note that as  $\Lambda$  increases above  $\Lambda=3$  then  $\zeta_{\min}$  steadily decreases and  $\zeta_{\max}$  steadily increases so that oscillations around  $\zeta = 1$  have a greater and greater "reach" on both sides of  $\zeta = 1$  with  $(1 - \zeta_{\min})$  and  $(\zeta_{\max} - 1)$  both increasing with increasing  $\Lambda$ .

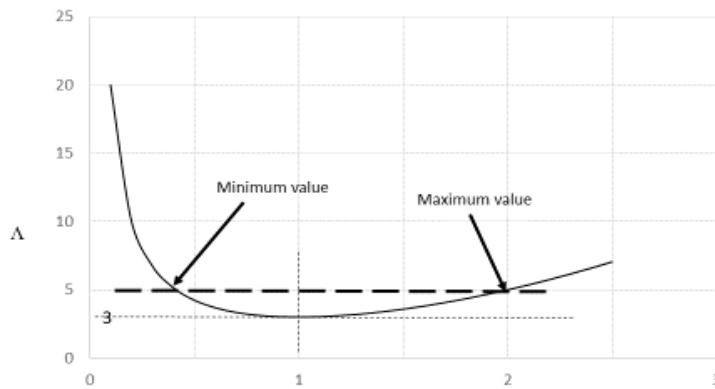


Figure 2. Normalized electron radius: equilibrium at value  $\zeta = 1$

Figure 2

Accordingly the normalized time  $T$  for an oscillation to vary from  $\zeta_{\min}$  to  $\zeta_{\max}$  and back is given by

$$T/2 = \int_{\zeta_{\min}}^{\zeta_{\max}} \frac{d\zeta(\zeta\Lambda - 2)}{[(\zeta\Lambda - 2)^2 - \zeta^6]^{1/2}} \quad (7)$$

where the limits of integration run from  $\zeta_{\min} < \zeta < \zeta_{\max}$ . The square root singularity in the integrand is accommodated in the customary manner so that for instance  $\int f(x)dx/(x-1)^{1/2}$  can be written proportional to  $f(x)(x-1)^{1/2} - \int (df(x)/dx)dx(x-1)^{1/2}$  and the integral then completed without concern for the singularity.

Of interest is to determine not only the total time  $T$  for an oscillation but equally the fraction of time spent at values of  $\zeta$  less than (greater than) unity for specific values of the constant  $\Lambda$ . The various panels of Figure 3 show the oscillation time as a function of increasing  $\Lambda$  (panel 3a), the relative percentage time the electron radius lies below the equilibrium value of  $\zeta = 1$  for increasing  $\Lambda$  (panel 3b), and the fraction of time the electron radius is greater than (less than)  $\zeta = 1$  as  $\Lambda$  increases (all values of  $\Lambda > 3$ ) (panels 3c and 3d respectively).

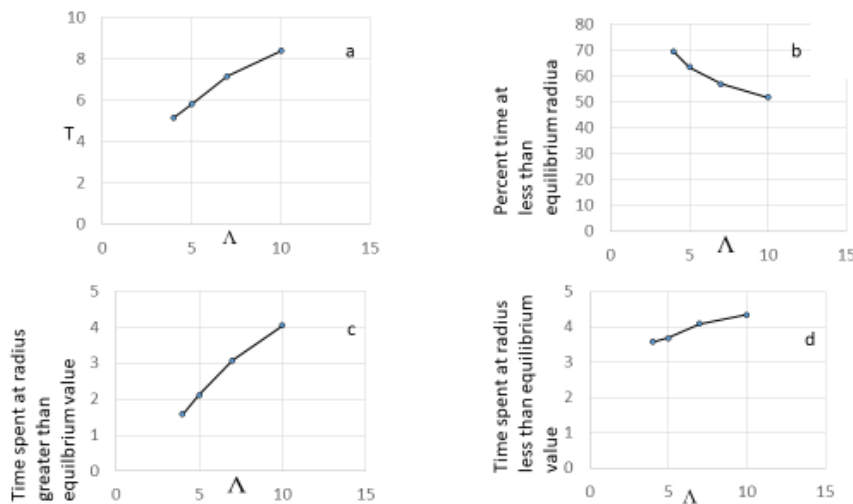


Figure 3

Correspondingly, Figure 4 shows the values of  $\zeta_{\min}$  and  $\zeta_{\max}$  for increasing values of  $\Lambda$ . Figure 5 provides a sketch of the oscillation time versus the reach showing an almost linear relationship between the pair with the implication that, on average, the speed of motion of the radius is almost constant no matter what values of the parameter  $\Lambda$  are used (at least as high as  $\Lambda = 10$  and likely higher).

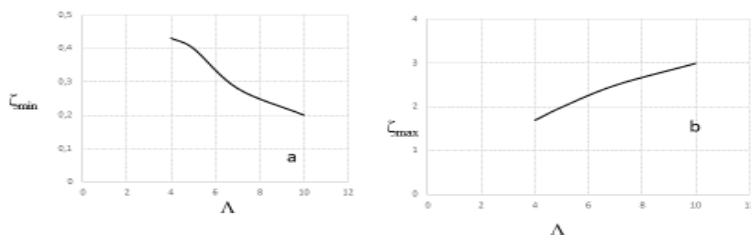


Figure 4

However such a linearity does not mean the speed is constant throughout every part of a complete cycle as can be seen in figure 3 where the time spent at values less than (greater than)the equilibrium value is sensitive to the  $\Lambda$  value chosen.

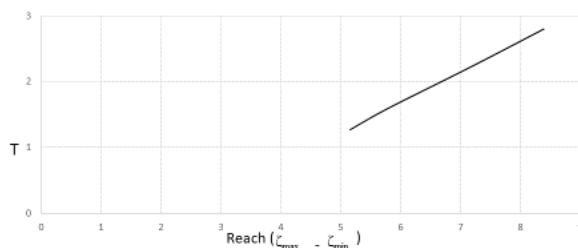


Figure 5

### A quantum representation

The classical theory of motion given above describes the exact solutions to the radial oscillation equation with just one “free” parameter  $\Lambda$ . There is no further constraint on the parameter without invoking behaviors not included directly in the equation of motion. Such is the purpose of the quantum representation. Two factors are needed. First a procedure for determining  $\Lambda$  in terms of a series of quantum states; second a procedure for determining the equivalent mass of a particle causing the oscillations once  $\Lambda$  has been prescribed.

#### a. Quantum representation of $\Lambda$ .

The Bohr-Sommerfeld quantization representation takes the classical electron momentum  $p (= mv/(1-v^2/c^2)^{1/2})$  and integrates it with respect to radius over the oscillation between the limits  $a_{\zeta_{\min}}$  to  $a_{\zeta_{\max}}$  and then demands that the integral be an integer multiple of Planck’s constant  $h$ :

$$\int_{a_{\zeta_{\min}}}^{a_{\zeta_{\max}}} p dr = nh \tag{8}$$

Given that the only free parameter is  $\Lambda$ , equation (8) provides a discrete series of values for  $\Lambda$  as the integer  $n$  is increased. Note that  $n=0$  describes an electron at rest at the radius  $a$  so that all other values correspond to excited states with increasing  $n$ .

*b. The equivalent particle mass, M*

Supposing that one has obtained the discrete values of  $\Lambda$  from equation (8) one can then determine the oscillation frequency  $\nu = c/(aT)$ , where  $T$  is the normalized dimensionless oscillation period given through equation (7). Then the equivalent mass of the particle causing the oscillation is  $M = h\nu/c^2$ . Collecting factors one can write

$$M/m = 2\pi(r_*/a) * ((\alpha T)^{-1}) \tag{9}$$

Because the oscillation period depends on  $\Lambda$  which is known from the quantization requirement (8) and because there are no further “free” parameters then the equivalent particle mass is given for each integer value of  $n > 0$ . However, note that if the normalized time,  $T$ , exceeds  $2\pi/\alpha$  then one would have  $M < m$  implying no solution.

This last part of the puzzle completes the Dirac prescription under the Bohr-Sommerfeld quantization conditions.

*c. Alternative quantum description*

There is no compelling reason that one should attempt to evaluate the parameter  $\Lambda$  first from the angular momentum quantization rule (8) and then attempt to deduce the equivalent mass from equation (9) using the value of  $\Lambda$  obtained from equation (8).

Instead one can determine the equivalent mass first and then use that value to attempt to determine the angular momentum constraint (8). The point is that the equation of motion is classical in origin and the quantization conditions are imposed ad hoc and are not an integral part of the development. Thus there is no good reason to suppose that values of  $\Lambda$  obtained from the angular momentum balance will also provide an equivalent mass satisfying the muon to electron mass ratio, and vice versa of course. Serendipity may prevail but one cannot assume it will. This point is now addressed quantitatively.

*d. Numerical illustrations*

Technically the angular momentum quantization of  $\Lambda$  takes the explicit form

$$\int_{\zeta_{\min}}^{\zeta_{\max}} [(\zeta\Lambda - 2)^2 - \zeta^6]^{1/2} \zeta^{-3} d\zeta = n\pi(r_*/a)/\alpha \tag{10}$$

where  $\alpha$  is the fine structure constant and so  $\pi/\alpha = 430.4$ . Because the classical electron radius  $r_*$  and the equilibrium radius  $a$  can be arranged to be equal the right hand side of equation (9) is large compared to unity for  $n > 0$ . Equation (10) allows determination of the discrete values of  $\Lambda$  as  $n$  is systematically increased. However there is a finite number of “eigenvalues” for  $\Lambda$  with increasing  $n$  because using equation (10) to find acceptable values of  $\Lambda$  implies that when the values are used in equation (7) the oscillation time will eventually exceed  $2\pi/\alpha$  so that the oscillations would, if existent, require  $M < m$  which is not allowed physically.

The evaluation of  $\Lambda$  eigenvalues satisfying the Bohr-Sommerfeld condition of equation (10) is complex because of the high degree of nonlinearity involving not only  $\Lambda$  directly in equation (10) but also through the dependence of the integral limits on  $\Lambda$ . It is not at all obvious that one obtains increasing  $\Lambda$  values as  $n$  is increased.

Simplification can be obtained although it would seem that the integrals for momentum and oscillation time must still be done numerically.

Following Richards (1959) define an auxiliary parameter  $\phi$  and determine  $\Lambda$  from  $\phi$  by

$$\cos\phi = -(3/\Lambda)^{3/2} \tag{11}$$

So as  $\Lambda$  varies continuously from 3 to  $\infty$ ,  $\phi$  decreases monotonically from  $\pi$  to  $\pi/2$ .

Then the three real roots of  $\zeta^3 - \Lambda\zeta + 2 = 0$  are ( $k=1, 2, 3$ )

$$\zeta_k = 2(\Lambda/3)^{1/2} \cos((\phi + 2\pi k)/3) \tag{12}$$

with  $k=2$  and  $k=3$  providing positive values.

One can then use equation (12) to simplify numerical methods. In his article Dirac (1962) concentrated on only the oscillation quantization around the equilibrium state and is silent concerning the angular momentum quantization. The full nonlinear solution to the equation of motion, while allowing one to investigate both factors, leads to a dichotomy as is now shown.

Using just a simple Excel integration method (which could be improved of course should one consider it necessary) one first determines the value of  $M/m$  provided by the integral given in equation (7) and with the aid of equation (9). More precisely one determines the value of  $\Lambda$  in equation (7) that will yield the ratio  $M/m = 206.8$ . The numerical result is  $M/m = 206.9$  with a corresponding value  $\Lambda = 4.84$ .

However when this now fixed value for  $\Lambda$  is inserted into the equation describing balance of angular momentum one obtains a value of 3.15 which is not even close to the value 430.4 required by the Bohr-Sommerfeld quantization.

The converse procedure is also available. Start with the angular momentum quantization and determine the value of  $\Lambda$  that will yield the required value of 430.4. Numerically the simple Excel program yields a best

value of 431.2 with  $\Lambda = 47.2$ . When this fixed value is now substituted in the oscillation quantization demand one obtains  $M/m = 51.6$  which is seemingly far away from the muon to electron mass ratio of 206.8.

In short: with just one of the two quantization conditions one can obtain a remarkably accurate fit to the measured values of 403.4 (angular momentum) or 206.8 (mass ratio for vibration frequency). However when both quantization conditions are invoked then there is, apparently, no common ground for a half way decent fit to both.

### Summary

While the primary purpose of this paper was to provide the general solution to the equation describing radial motion for a classical extensible model of an electron as formulated by Dirac(1962), it was also possible to use the general solution in a Bohr-Sommerfeld quantization scheme for both angular momentum and vibrations. One cannot satisfy both quantum requirements simultaneously but one can satisfy one or the other. Dirac considered only the vibration quantization in his article and was silent about the angular momentum requirement.

Ignoring electron spin, radiation from vibrations and dealing with a classical electron motion means that it is not entirely unexpected that one does not reproduce the observed behavior of the electron, muon and, perhaps, tau. It is encouraging that the nonlinear solution to the classical equation does allow one to reproduce exactly one of the quantization requirements. While the results given are a long way from providing a complete theory of electron behavior they do indicate that one should be more open to general solutions in attempts to describe electron extensibility.

### Acknowledgement

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### Figure Captions

Figure 1. Normalized electron radius: equilibrium at value  $\zeta = 1$ .

Figure 2. Normalized electron radius sketching the “turning points” for localized behavior.

Figure 3. The oscillation time as a function of increasing  $\Lambda$  (panel 3a), the relative percentage of time the electron radius lies below the equilibrium value of  $\zeta = 1$  for increasing  $\Lambda$  (panel 3b), and the fraction of time the electron radius is greater than (less than)  $\zeta = 1$  as  $\Lambda$  increases (all values of  $\Lambda > 3$ , panels 3c and 3d respectively).

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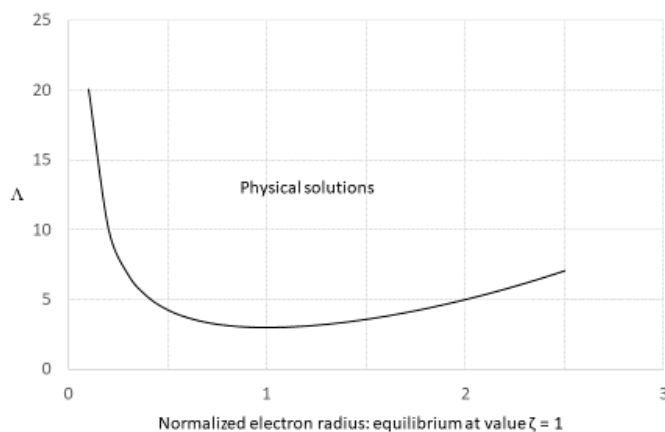


Figure 1

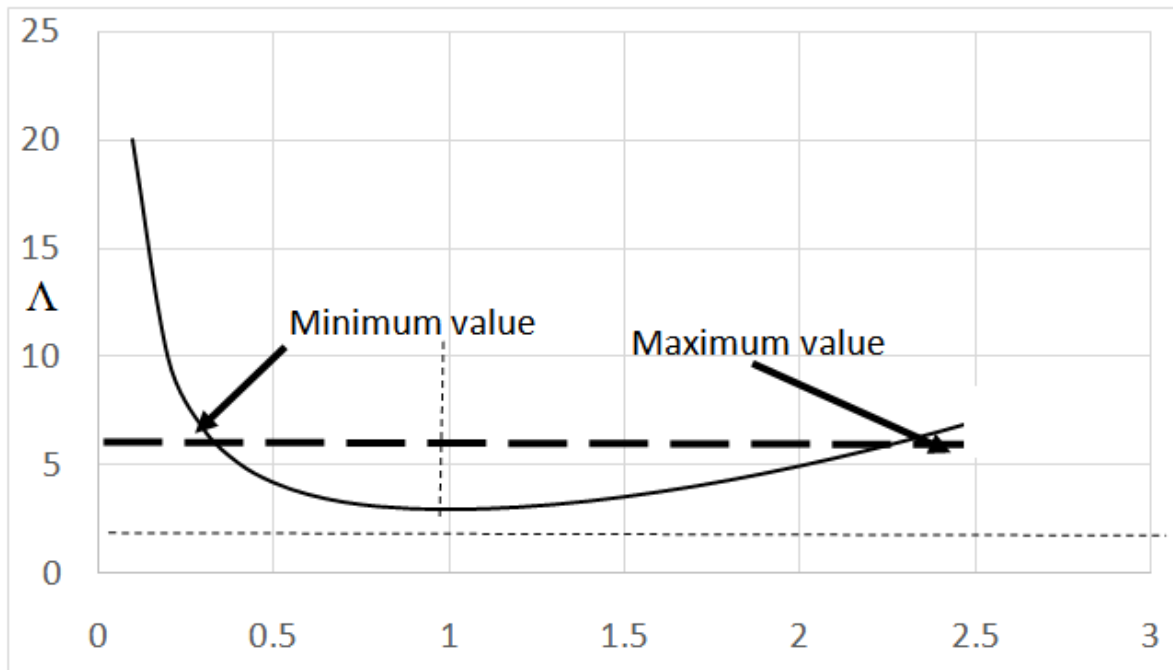


Figure 2. Normalized electron radius: equilibrium at value  $\zeta = 1$

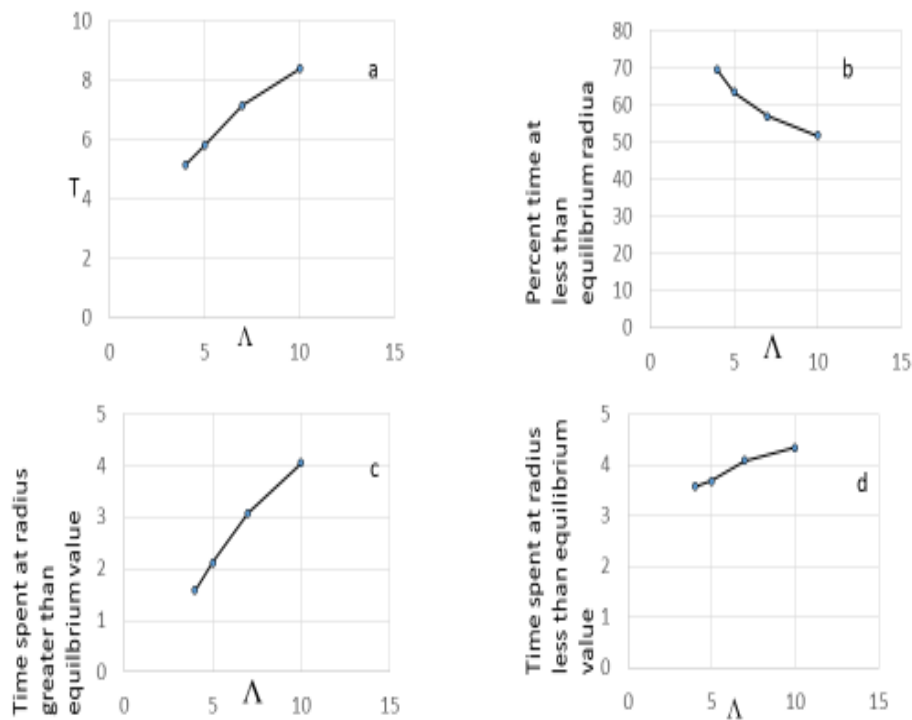


Figure 3

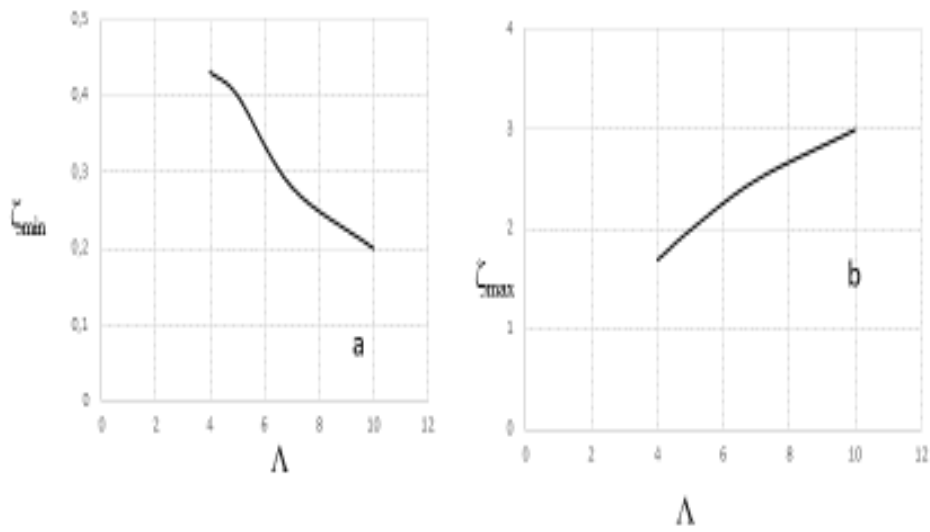


Figure 4

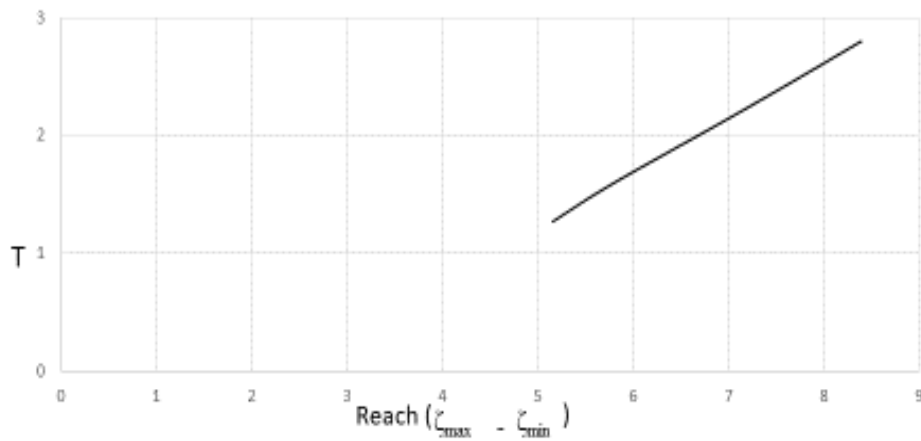


Figure 5

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